Modal Logic: Overview

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LMU, The Modal Logic Sessions

The Basics

The first session

- Modal logics: syntax and semantics
- Invariance results: for K_n, the class of models is restricted to finite trees.
- Decidability: PSPACE-complete

Calculi for Modal Logics

The second session

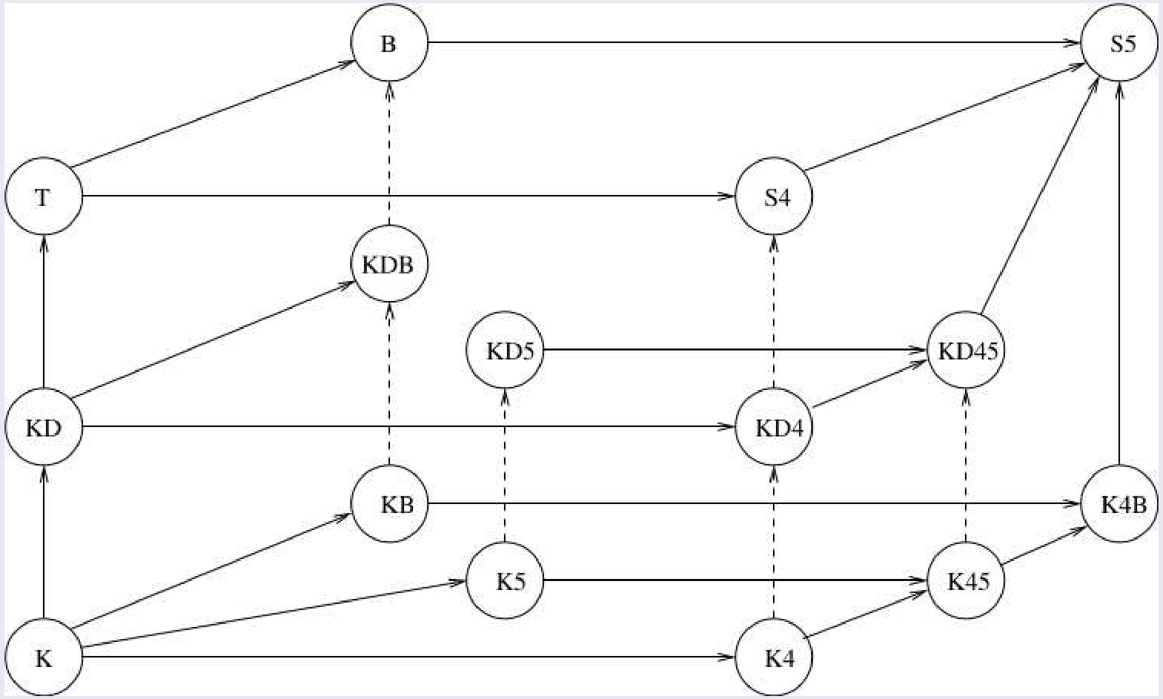
- Axiomatisations for monomodal logic K_1
- Prefixed tableaux for monomodal logic K_1
- How these are extended to deal with multimodal logics K_n

Extensions

Some Other Usual Modal Logics

Different restrictions on the accessibility relations $\mathcal{R}_a$ define different modal logics:

- No restrictions: K_n;
- Reflexive: KT_n;
- Transitive: $K4_n$;
- Euclidean: $K5_n$;
- Serial: KD_n;
- Symmetric: KB_n;
- Reflexive and Transitive: $S4_n$;
- Reflexive and Euclidean: $S5_n$;
-



K4

- This is the logic related to transitive relations:
  - Syntax is the same as before.
  - Semantics is given by a Kripke Structure $\mathcal{M}$ for $\mathcal{P}$ and $\mathcal{A} = \{1, \ldots, n\}$ is a tuple

$$\mathcal{M} = \langle \mathcal{W}, \mathcal{R}_1, \ldots, \mathcal{R}_n, \pi \rangle,$$

    where:
    - $\mathcal{W}$ is a non-empty set;
    - For each $a \in \mathcal{A}$, $\mathcal{R}_a \subseteq \mathcal{W} \times \mathcal{W}$, where each $\mathcal{R}_a$ is transitive;
    - $\pi : \mathcal{W} \times \mathcal{P} \longrightarrow \{T, F\}$.
  - The satisfiability relation is defined as before.
  - The notions of satisfiability/validity of a formula are defined exactly as before.

Frame Characterisation and Axioms

Name	Axiom	Frame Property	
D	$\Box_a \varphi \rightarrow \Diamond_a \varphi$	Serial	$\forall v \exists w . \mathcal{R}_a vw$
T	$\Box_a \varphi \rightarrow \varphi$	Reflexive	$\forall w . \mathcal{R}_a ww$
B	$\varphi \rightarrow \Box_a \Diamond_a \varphi$	Symmetric	$\forall vw . \mathcal{R}_a vw \rightarrow \mathcal{R}_a wv$
4	$\Box_a \varphi \rightarrow \Box_a \Box_a \varphi$	Transitive	$\forall uvw . (\mathcal{R}_a uv \wedge \mathcal{R}_a vw) \rightarrow \mathcal{R}_a uw$
5	$\Diamond_a \varphi \rightarrow \Box_a \Diamond_a \varphi$	Euclidean	$\forall uvw . (\mathcal{R}_a uv \wedge \mathcal{R}_a uw) \rightarrow \mathcal{R}_a vw$

Axiomatisation

Taut	enough propositional tautologies.
K	$\boxed{a}(\varphi \to \psi) \to (\boxed{a}\varphi \to \boxed{a}\psi)$.
SUB	Uniform substitution; and
MP	If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$.
Nec	If $\vdash \varphi$, then $\vdash \boxed{a}\varphi$

You can also add:

Dual	$\Diamond_a \varphi \leftrightarrow \neg \boxed{a} \neg \varphi$

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B	$\varphi \to \boxed{a}\Diamond_a\varphi$

and

SUB	Uniform substitution; and
MP	If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$.
Nec	If $\vdash \varphi$, then $\vdash \boxed{a}\varphi$

You can also add:

Dual	$\Diamond_a\varphi \leftrightarrow \neg\boxed{a}\neg\varphi$

Example

We want a proof for 5 ($\Diamond_a p \to \Box_a \Diamond_a p$) in the system containing the axioms K, B, and 4. In the following,

- K : $\Box_a(\varphi \to \psi) \to (\Box_a \varphi \to \Box_a \psi)$
- B : $\varphi \to \Box_a \Diamond_a \varphi$
- 4 : $\Box_a \varphi \to \Box_a \Box_a a\varphi$
- *chaining*: $((\varphi \to \psi) \land (\psi \to \chi)) \to (\varphi \to \chi)$.

1.	$\Box_a \neg p \to \Box_a \Box_a \neg p$	[4, $\varphi = \neg p$]
2.	$\Diamond_a \Diamond_a p \to \Diamond_a p$	[contrapositive,1]
3.	$\Box_a(\Diamond_a \Diamond_a p \to \Diamond_a p)$	[NEC, 2]
4.	$\Box_a(\Diamond_a \Diamond_a p \to \Diamond_a p) \to (\Box_a \Diamond_a \Diamond_a p \to \Box_a \Diamond_a p)$	[K, $\varphi = \Diamond_a \Diamond_a p$, $\psi = \Diamond_a p$]
5.	$\Box_a \Diamond_a \Diamond_a p \to \Box_a \Diamond_a p$	[MP,3,4]
6.	$\Diamond_a p \to \Box_a \Diamond_a \Diamond_a p$	[B, $\varphi = \Diamond_a p$]
7.	$\Diamond_a p \to \Box_a \Diamond_a p$	[Chaining,6,5]

Note that because we have uniform substitution, replacing p in the conclusion gives us all instances of 5.

Tableaux

α	β	γ	δ
$\dfrac{\sigma : \varphi \wedge \psi}{\begin{array}{c}\sigma : \varphi \\ \sigma : \psi\end{array}}$	$\dfrac{\sigma : \varphi \vee \psi}{\sigma : \varphi \mid \sigma : \psi}$	$\dfrac{\sigma : \Box\varphi}{\sigma.i : \varphi}$ for all existing $\sigma.i$	$\dfrac{\sigma : \Diamond\varphi}{\sigma.i : \varphi}$ for a fresh $\sigma.i$

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$\dfrac{\sigma : \varphi \wedge \psi}{\begin{array}{c}\sigma : \varphi \\ \sigma : \psi\end{array}}$	$\dfrac{\sigma : \varphi \vee \psi}{\sigma : \varphi \mid \sigma : \psi}$	$\dfrac{\sigma : \Box\varphi}{\sigma.i : \varphi}$ for all existing $\sigma.i$	$\dfrac{\sigma : \Diamond\varphi}{\sigma.i : \varphi}$ for a fresh $\sigma.i$

T	D	B	4	$4r$
$\Box\varphi \to \varphi$	$\Box\varphi \to \Diamond\varphi$	$\varphi \to \Box\Diamond\varphi$	$\Box\varphi \to \Box\Box\varphi$	
$\dfrac{\sigma : \Box\varphi}{\sigma : \varphi}$	$\dfrac{\sigma : \Box\varphi}{\sigma : \Diamond\varphi}$	$\dfrac{\sigma.i : \Box\varphi}{\sigma : \varphi}$	$\dfrac{\sigma : \Box\varphi}{\sigma.i : \Box\varphi}$ for all existing $\sigma.i$	$\dfrac{\sigma.i : \Box\varphi}{\sigma : \Box\varphi}$

Note: For S5 we have (T+4+4r).

Example: $(\Box p \wedge \Box q) \rightarrow \Box(\Box p \wedge \Box q)$

(1) 1: $(\Box p \wedge \Box q) \wedge \Diamond(\Diamond \neg p \vee \Diamond \neg q)$ [neg. assumption]

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Termination in K4-Tableaux

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$$\Box p \lor \Diamond\Box p$$

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Resolution

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- The calculus for classical logic can be adapted to non-classical logics. Those calculi are also reasonably simple and they are also efficient in practice (or, to be more precise, provers are comparable to state-of-art provers implementing other calculi).

Clausal Resolution for Propositional Logic

- Resolution is a refutational procedure: if we want to prove φ, we apply the rules to $\neg\varphi$.
- Reasoning is performed backwards: from what we want to prove to axioms (in this case, a contradiction).
- Resolution can be clausal or non-clausal: for clausal resolution, we transform $\neg\varphi$ into CNF before applying the inference rule.
- There is only one inference rule:

$$[\text{RES}] \quad \frac{\begin{array}{ccr} (\varphi & \vee & l) \\ (\psi & \vee & \neg l) \end{array}}{\begin{array}{ccr} (\varphi & \vee & \psi) \end{array}}$$

- Premises are called parent clauses (or the resolvends). The conclusion is called the resolvent. The literals l and $\neg l$ are known as complementary literals. The parent clauses are *resolved* on the complementary literals, generating the resolvent.
- The inference rule is applied until either a contradiction is found or no new clauses can be generated.

Procedure

Let Γ_0 be a set of clauses.

$i \leftarrow 0$
repeat
    Choose c_1 and $c_2 \in \Gamma_i$ such that $l \in c_1$ and $\neg l \in c_2$
    Calculate the resolvent r
    if r is not redundant **then**
        Let $\Gamma_{i+1} \leftarrow \Gamma_i \cup \{r\}$
    end if
    $i \leftarrow i + 1$
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repeat is preceded by $i \leftarrow 0$:

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CNF

- A literal is a propositional symbol or its negation.
- A clause is a disjunction of literals.
- A formula of the form

$$\bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} l_{ij}$$

is in conjunctive normal form (CNF).

Let $\varphi \in$ WFF.
There is $\varphi' \in$ WFF, $\varphi' \dashv\models \varphi$ and φ' is in CNF.

The usual rewritring rules

- Combinatorial explosion:

$$(l_1 \wedge \ldots \wedge l_m) \vee (k_1 \wedge \ldots \wedge k_n) \Rightarrow O(m \times n)$$

- In general:

ψ	$w(\psi)$	$\overline{w}(\psi)$
$\varphi_1 \wedge \ldots \wedge \varphi_n$	$\sum_{i=1}^{n} w(\varphi_i)$	$\prod_{i=1}^{n} \overline{w}(\varphi_i)$
$\varphi_1 \vee \ldots \vee \varphi_n$	$\prod_{i=1}^{n} w(\varphi_i)$	$\sum_{i=1}^{n} \overline{w}(\varphi_i)$
$\varphi_1 \rightarrow \varphi_2$	$\overline{w}(\varphi_1)w(\varphi_2)$	$w(\varphi_1) + \overline{w}(\varphi_2)$
$\varphi_1 \leftrightarrow \varphi_2$	$w(\varphi_1)\overline{w}(\varphi_2) + \overline{w}(\varphi_1)w(\varphi_2)$	$w(\varphi_1)w(\varphi_2) + \overline{w}(\varphi_1)\overline{w}(\varphi_2)$
$\neg\varphi$	$\overline{w}(\varphi)$	$w(\varphi)$
atomic	1	1

Renaming

- We introduce new literals which replace subformulae;
- We also need to introduce the *definition* clauses for those literals.
  Let φ be the formula to be replaced:

$$
\begin{array}{lcl}
Pol(\varphi) > 0 & \Rightarrow & new_{\varphi} \rightarrow \varphi \\
Pol(\varphi) < 0 & \Rightarrow & \varphi \rightarrow new_{\varphi} \\
Pol(\varphi) = 0 & \Rightarrow & new_{\varphi} \leftrightarrow \varphi
\end{array}
$$

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$$\begin{array}{rcl} Pol(\varphi) > 0 & \Rightarrow & new_\varphi \rightarrow \varphi \\ Pol(\varphi) < 0 & \Rightarrow & \varphi \rightarrow new_\varphi \\ Pol(\varphi) = 0 & \Rightarrow & new_\varphi \leftrightarrow \varphi \end{array}$$

Let $\varphi \in$ WFF. There is $\varphi' \in$ WFF, φ' is in CNF, and φ' is satisfiable if, and only if, φ is satisfiable. Moreover, size$(\varphi') = O($size$(\varphi))$.

Example

$$(a \wedge b \wedge f) \vee (c \wedge d \wedge e)$$

Example

$$(a \wedge b \wedge f) \vee (c \wedge d \wedge e)$$

$$new_{(a \wedge b \wedge f)} \vee new_{(c \wedge d \wedge e)}$$

$$new_{(a \wedge b \wedge f)} \rightarrow (a \wedge b \wedge f) \quad new_{(c \wedge d \wedge e)} \rightarrow (c \wedge d \wedge e)$$

Example

$$(a \wedge b \wedge f) \vee (c \wedge d \wedge e)$$

$$new_{(a \wedge b \wedge f)} \vee new_{(c \wedge d \wedge e)}$$

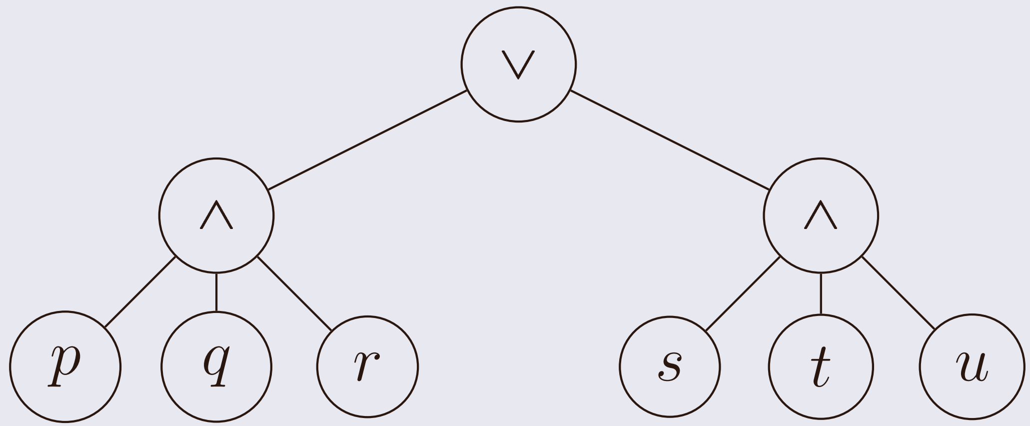
$$new_{(a \wedge b \wedge f)} \rightarrow (a \wedge b \wedge f) \quad new_{(c \wedge d \wedge e)} \rightarrow (c \wedge d \wedge e)$$

$$\neg new_{(a \wedge b \wedge f)} \vee a \qquad \neg new_{(c \wedge d \wedge e)} \vee c$$

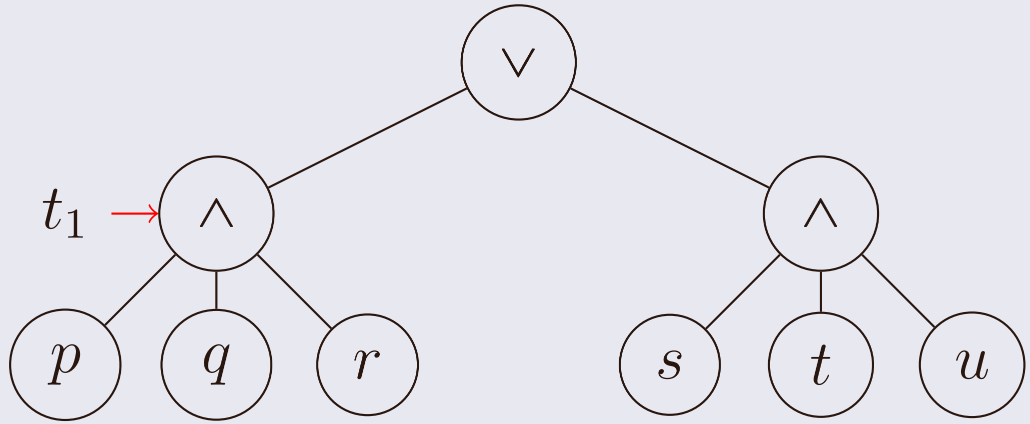
$$\neg new_{(a \wedge b \wedge f)} \vee b \qquad \neg new_{(c \wedge d \wedge e)} \vee d$$

$$\neg new_{(a \wedge b \wedge f)} \vee f \qquad \neg new_{(c \wedge d \wedge e)} \vee e$$

Example



Example



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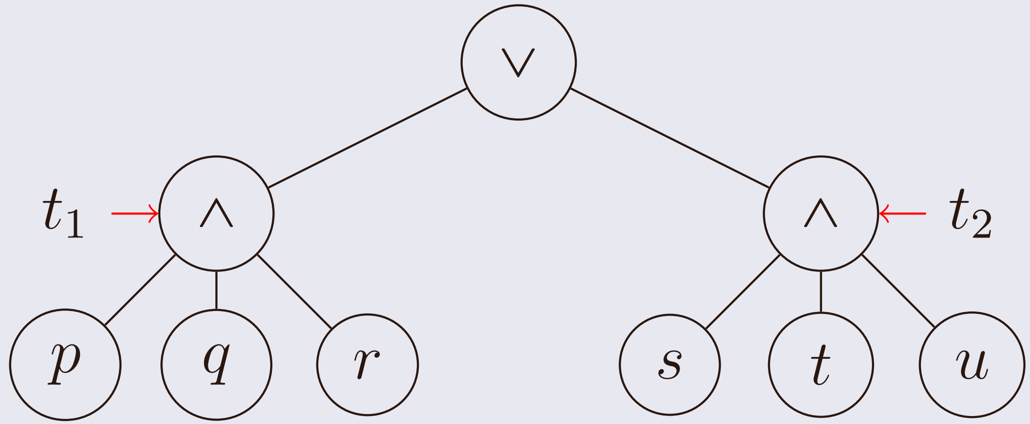

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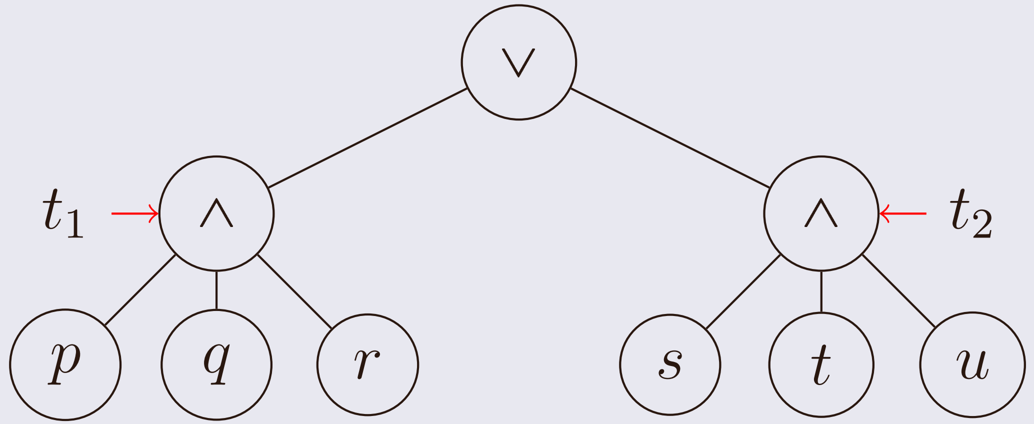

$$(t_1 \vee t_2) \wedge (t_1 \rightarrow p \wedge q \wedge r) \wedge (t_2 \rightarrow s \wedge t \wedge u)$$

$$(t_1 \vee t_2) \wedge (t_1 \vee p) \wedge (t_1 \vee q) \wedge (t_1 \vee r) \wedge (t_2 \vee s) \wedge (t_2 \vee t) \wedge (t_2 \vee u)$$

More on Renaming

- Renaming ensures that the CNF of a formula has size linear on the size of that formula.
- Renaming helps separating different contexts for reasoning:

$$t \longrightarrow \Diamond_1 \Diamond_2 p$$

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Resolution-based Calculi for Modal Logics

There are some few:

- Fitting's destructive resolution for modal logics [Fit90]
- Mint's resolution for modal logics [Mints, 1990]
- Fariñas de Cerro and colleagues [Far82, dCH88, EdC89, dCH90]
- Non-clausal resolution by Abadi & Manna [AM86]

GMR

This was introduced in [ND, 2007].

- We came up with the normal form while investigating techniques for preprocessing of formulae (prenex and antiprenexing) [ND, 2006]
- The calculus is now referred to as GMR for Global Modal Resolution, as it uses global renaming.
- It is quite simple, the propositional and the modal part are completely separated.
- The (truly) modal rules are hyper-rules.

The Normal Form

In [ND, 2006, ND, 2007]:

- Initial clause $\boxed{*}(\mathrm{start} \to \bigvee_{b=1}^{r} l_b)$
- Literal clause $\boxed{*}(\mathrm{true} \to \bigvee_{b=1}^{r} l_b)$
- Positive a-clause $\boxed{*}(l' \to \boxed{a} l)$
- Negative a-clause $\boxed{*}(l' \to \Diamond_a l)$

where l, l', $l_b \in \mathcal{L}$. Positive and negative a-clauses are together known as *modal* a*-clauses*; the index a may be omitted if it is clear from the context.

Classical Resolution

$$\text{[IRES1]}\quad \frac{\begin{array}{lcl} \boxed{*}(\mathbf{true} & \rightarrow & D \lor l) \\ \boxed{*}(\mathbf{start} & \rightarrow & D' \lor \neg l) \end{array}}{\begin{array}{lcl}\boxed{*}(\mathbf{start} & \rightarrow & D \lor D')\end{array}}$$

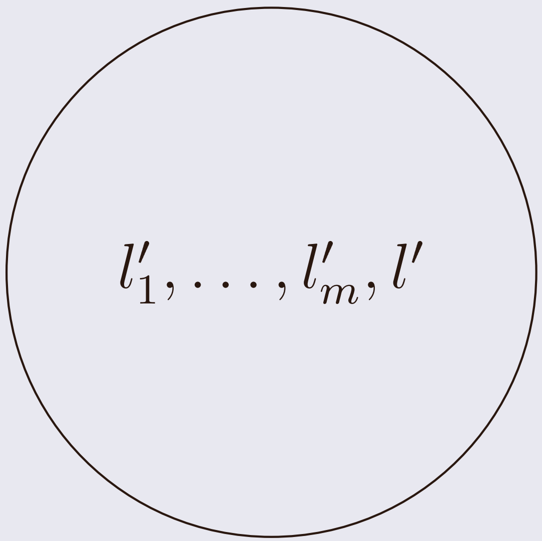
$$\text{[IRES2]}\quad \frac{\begin{array}{lcl} \boxed{*}(\mathbf{start} & \rightarrow & D \lor l) \\ \boxed{*}(\mathbf{start} & \rightarrow & D' \lor \neg l) \end{array}}{\begin{array}{lcl}\boxed{*}(\mathbf{start} & \rightarrow & D \lor D')\end{array}}$$

$$\text{[LRES]}\quad \frac{\begin{array}{lcl} \boxed{*}(\mathbf{true} & \rightarrow & D \lor l) \\ \boxed{*}(\mathbf{true} & \rightarrow & D' \lor \neg l) \end{array}}{\begin{array}{lcl}\boxed{*}(\mathbf{true} & \rightarrow & D \lor D')\end{array}}$$

$$\text{[MRES]}\quad \frac{\begin{array}{lcl} \boxed{*}(l_1 & \rightarrow & \boxed{a}\, l) \\ \boxed{*}(l_2 & \rightarrow & \Diamond_a \neg l) \end{array}}{\begin{array}{lcl}\boxed{*}(\mathbf{true} & \rightarrow & \neg l_1 \lor \neg l_2)\end{array}}$$

Modal Resolution I

$$
\text{[GEN1]}\quad
\begin{array}{rcl}
\boxed{*}(l'_1 & \rightarrow & \boxed{a}\neg l_1) \\
 & \vdots & \\
\boxed{*}(l'_m & \rightarrow & \boxed{a}\neg l_m) \\
\boxed{*}(l' & \rightarrow & \langle a \rangle \neg l) \\
\boxed{*}(\text{true} & \rightarrow & l_1 \vee \ldots \vee l_m \vee l) \\
\hline
\boxed{*}(\text{true} & \rightarrow & \neg l'_1 \vee \ldots \vee \neg l'_m \vee \neg l')
\end{array}
$$



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\hline
& \boxed{*}(\mathbf{true} & \rightarrow & \neg l'_1 \vee \ldots \vee \neg l'_m \vee \neg l')
\end{array}
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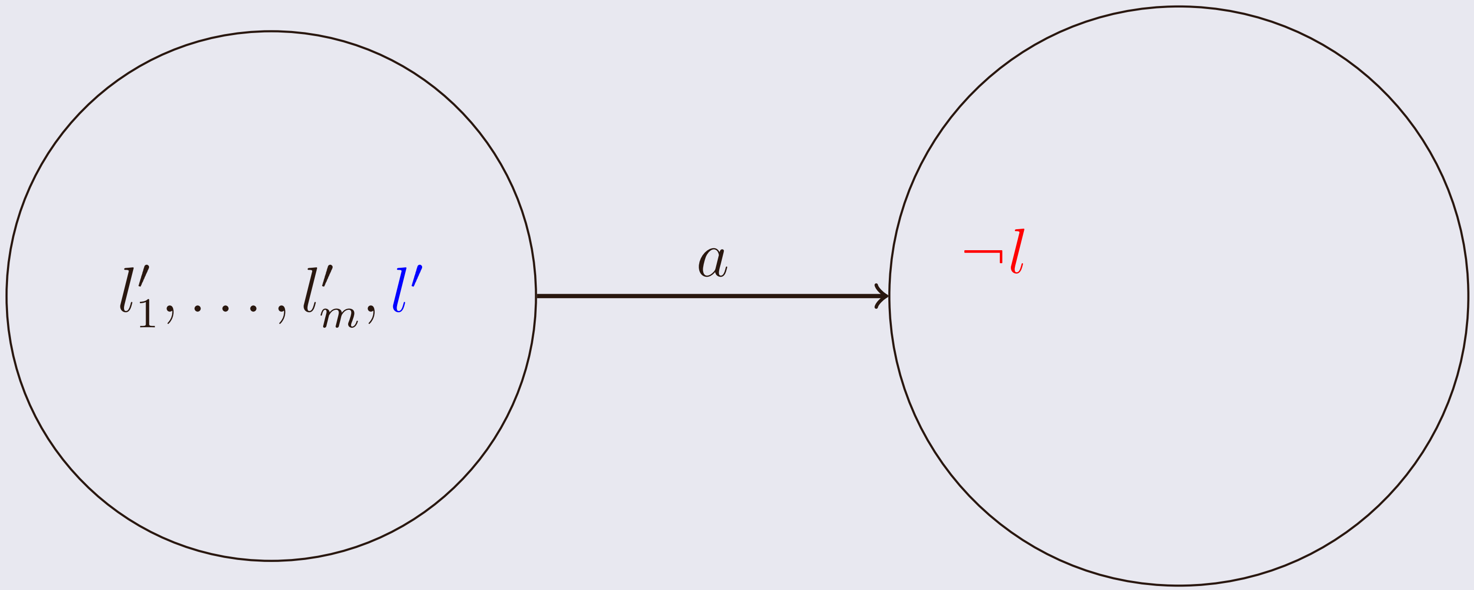

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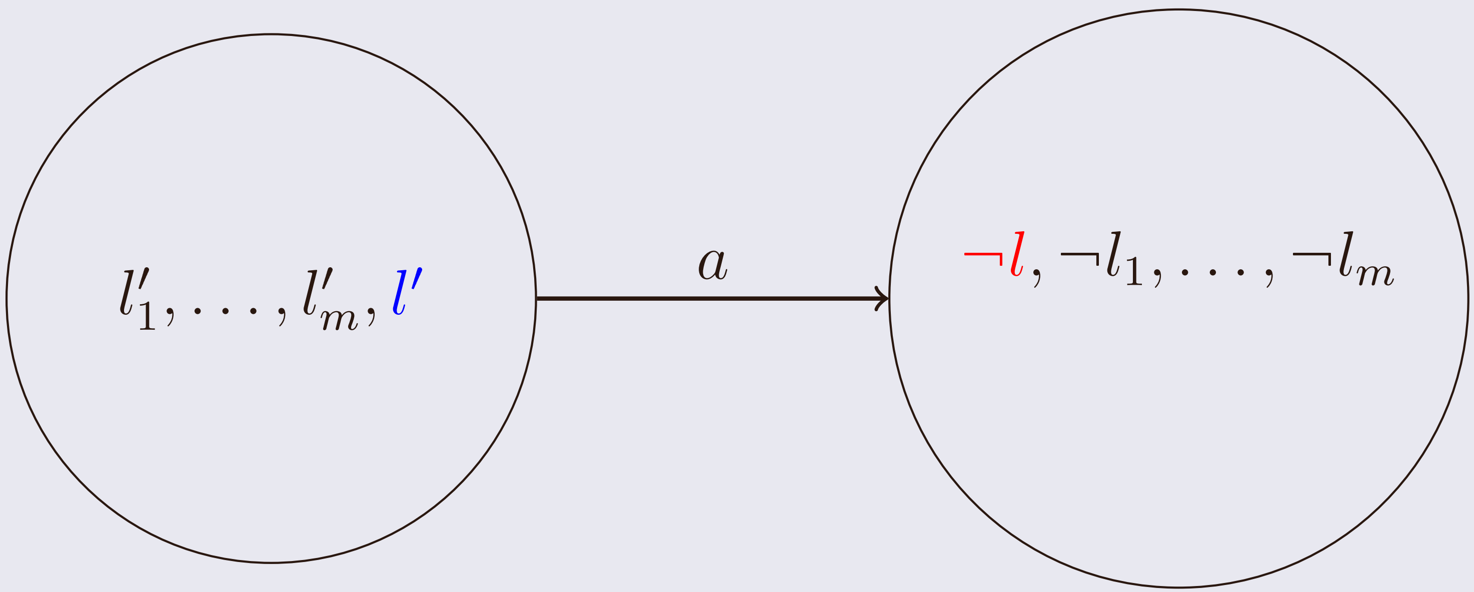
\end{array}
$$



Modal Resolution I

$$
\begin{array}{lrcl}
\text{[GEN1]} & \boxed{*}(l'_1 & \to & \boxed{a}\neg l_1) \\
 & & \vdots & \\
 & \boxed{*}(l'_m & \to & \boxed{a}\neg l_m) \\
 & \boxed{*}(l' & \to & \Diamond_a \neg l) \\
 & \boxed{*}(\mathbf{true} & \to & l_1 \vee \ldots \vee l_m \vee l) \\
\hline
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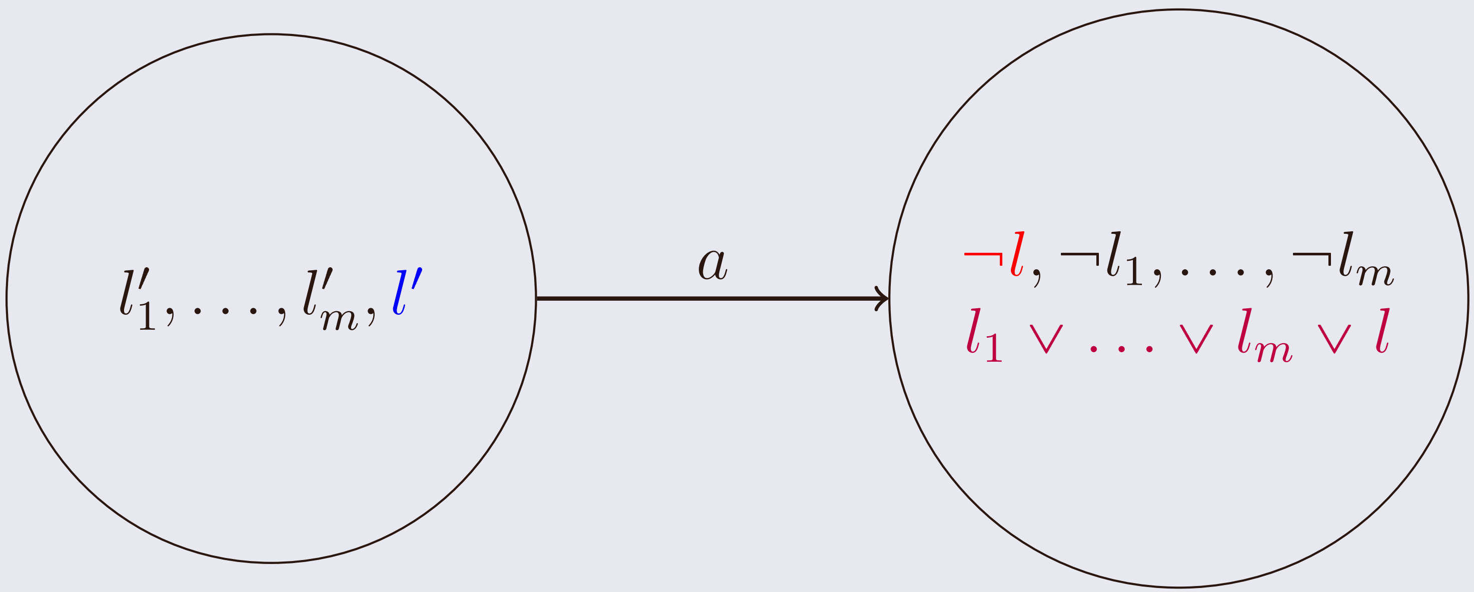
\end{array}
$$



Modal Resolution II

[GEN2]

$$\dfrac{\begin{array}{rcl} \boxed{*}(l'_1 & \rightarrow & \boxed{a}\, l_1) \\ \boxed{*}(l'_2 & \rightarrow & \boxed{a}\neg l_1) \\ \boxed{*}(l'_3 & \rightarrow & \Diamond_a\, l_2) \end{array}}{\boxed{*}(\mathrm{true} \rightarrow \neg l'_1 \vee \neg l'_2 \vee \neg l'_3)}$$

[GEN3]

$$\dfrac{\begin{array}{rcl} \boxed{*}(l'_1 & \rightarrow & \boxed{a}\neg l_1) \\ & \vdots & \\ \boxed{*}(l'_m & \rightarrow & \boxed{a}\neg l_m) \\ \boxed{*}(l' & \rightarrow & \Diamond_a\, l) \\ \boxed{*}(\mathrm{true} & \rightarrow & l_1 \vee \ldots \vee l_m) \end{array}}{\boxed{*}(\mathrm{true} \rightarrow \neg l'_1 \vee \ldots \vee \neg l'_m \vee \neg l')}$$

Example - NNF - I

$$\Box(p \to q) \to (\Box p \to \Box q)$$

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then we apply renaming to start the transformation:

$$\boxed{*}(\text{start} \to t_0) \wedge \boxed{*}(t_0 \to \Box(p \to q) \wedge \Box p \wedge \Diamond\neg q)$$

Example - NNF - II

$$\boxed{*}(\text{start} \to t_0) \wedge \boxed{*}(t_0 \to \Box(p \to q) \wedge \Box p \wedge \Diamond\neg q)$$

we apply rewriting and get:

$$\boxed{*}(\text{start} \to t_0), \boxed{*}(t_0 \to \Box(p \to q)), \boxed{*}(t_0 \to \Box p), \boxed{*}(t_0 \to \Diamond\neg q)$$

Example - NNF - II

$$\boxed{*}(\text{start} \rightarrow t_0) \wedge \boxed{*}(t_0 \rightarrow \Box(p \rightarrow q) \wedge \Box p \wedge \Diamond \neg q)$$

we apply rewriting and get:

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and we are almost there. We just need to apply renaming to the second formula above:

$$\boxed{*}(t_0 \rightarrow \Box t_1), \boxed{*}(t_1 \rightarrow (p \rightarrow q))$$

and rewrite it in the normal form. We obtain the following set of clauses:

$$\boxed{*}(\text{start} \rightarrow t_0)$$

$$\boxed{*}(t_0 \rightarrow \Box t_1)$$

$$\boxed{*}(\text{true} \rightarrow \neg t_1 \vee \neg p \vee q)$$

$$\boxed{*}(t_0 \rightarrow \Box p)$$

$$\boxed{*}(t_0 \rightarrow \Diamond \neg q)$$

A Refutation

1. $\boxed{*}(\text{start} \rightarrow t_0)$
2. $\boxed{*}(t_0 \rightarrow \Box t_1)$
3. $\boxed{*}(\text{true} \rightarrow \neg t_1 \vee \neg p \vee q)$
4. $\boxed{*}(t_0 \rightarrow \Box p)$
5. $\boxed{*}(t_0 \rightarrow \Diamond \neg q)$

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4. $\boxed{*}(t_0 \rightarrow \Box p)$
5. $\boxed{*}(t_0 \rightarrow \Diamond \neg q)$
6. $\boxed{*}(\mathbf{true} \rightarrow \neg t_0)$ $[\mathsf{GEN1}, 2, 4, 5, 3]$

A Refutation

1.	$\boxed{*}(\text{start} \rightarrow t_0)$	
2.	$\boxed{*}(t_0 \rightarrow \Box t_1)$	
3.	$\boxed{*}(\text{true} \rightarrow \neg t_1 \vee \neg p \vee q)$	
4.	$\boxed{*}(t_0 \rightarrow \Box p)$	
5.	$\boxed{*}(t_0 \rightarrow \Diamond \neg q)$	
6.	$\boxed{*}(\text{true} \rightarrow \neg t_0)$	$[\mathsf{GEN1}, 2, 4, 5, 3]$
7.	$\boxed{*}(\text{start} \rightarrow \text{false})$	$[\mathsf{IRES1}, 7, 1]$

Causes of Inefficiency

$$\Diamond\Diamond p \wedge \Box\neg p$$

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$$\Diamond\Diamond p \wedge \Box\neg p$$

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2. $\boxed{*}(t_0 \rightarrow \Diamond t_1)$
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Causes of Inefficiency

$\Diamond\Diamond p \land \Box\neg p$

1. $\text{start} \rightarrow t_0$
2. $\boxed{*}(t_0 \rightarrow \Diamond t_1)$
3. $\boxed{*}(t_1 \rightarrow \Diamond p)$ $\quad \neg\Box\neg p$
4. $\boxed{*}(t_0 \rightarrow \Box\neg p)$

$$\Diamond\Diamond p \wedge \Box\neg p$$

1. $\text{start} \rightarrow t_0$
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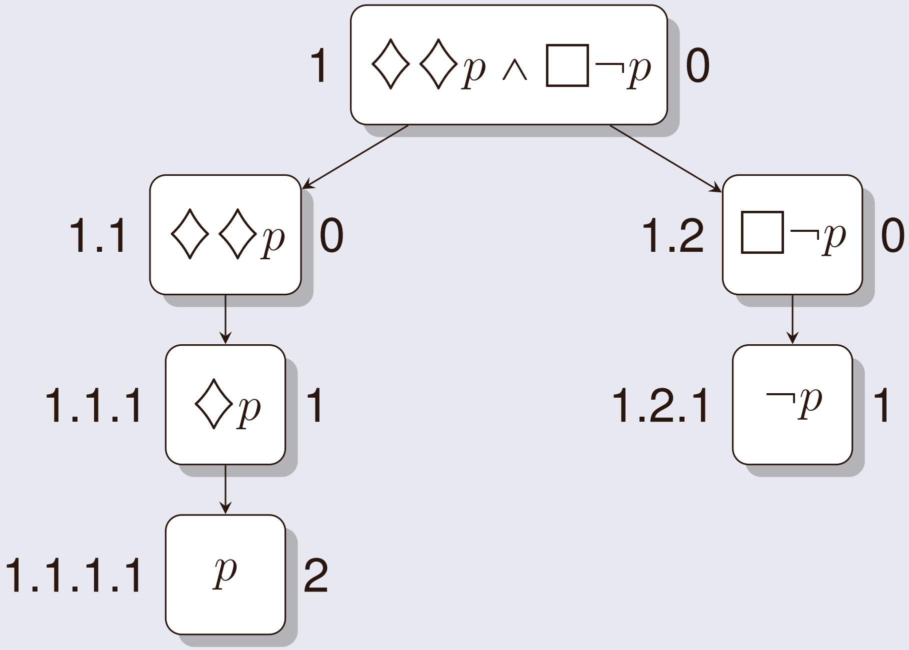
4. $\boxed{*}(t_0 \rightarrow \Box\neg p)$



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$$\Diamond\Diamond p \wedge \Box\neg p$$

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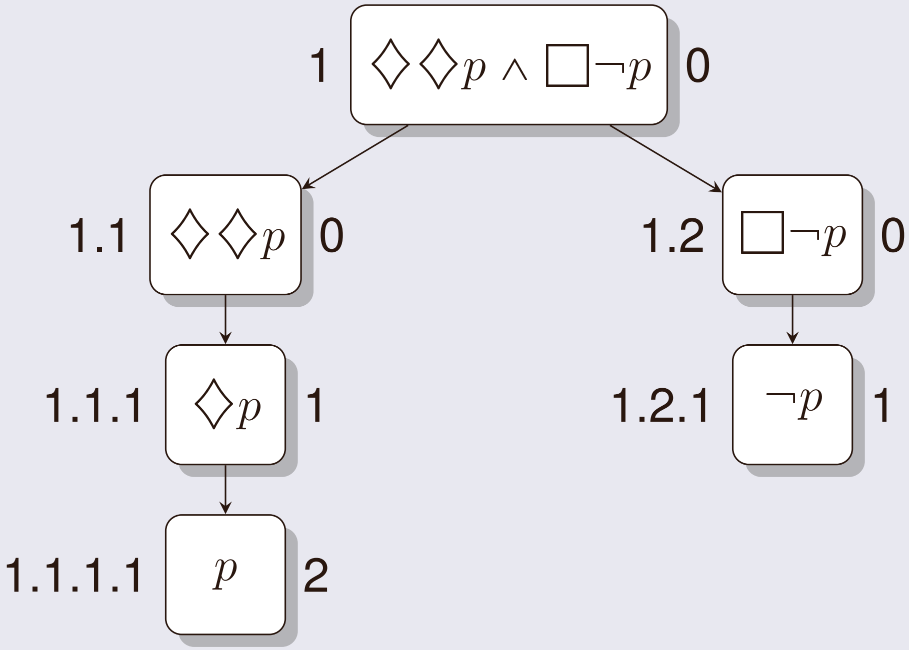

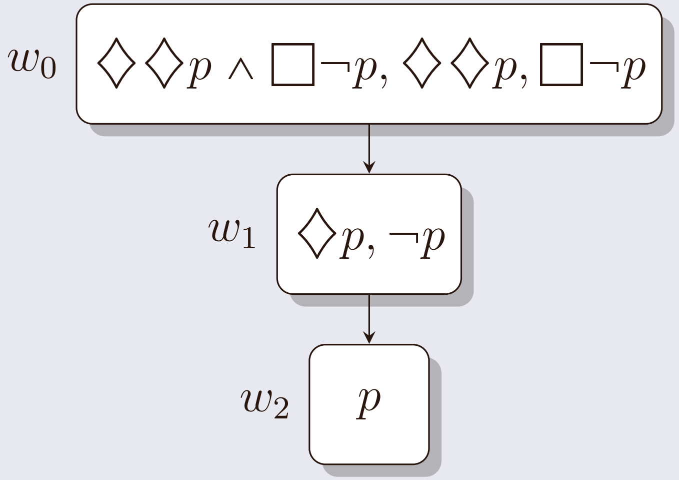

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