Modal Logic: Overview

Cláudia Nalon

Department of Computer Science

University of Brasília

LMU, The Modal Logic Sessions

The Basics

The last session

- Modal logics: syntax and semantics
- Invariance results: for K_n, the class of models is restricted to finite trees.
- Decidability: PSPACE-complete

Calculi for Modal Logics

Notation and Properties

- A calculus for a logic L is a pair $\langle \mathcal{A}, \mathcal{R} \rangle_L$, where $\mathcal{A} \subseteq \text{WFF}_L$ and $\mathcal{R} \subseteq (2^{\text{WFF}_L} \times \text{WFF}_L)$.

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- Let $\varphi \in \text{WFF}_L$. Let $\mathcal{C} = \langle \mathcal{A}, \mathcal{R} \rangle_L$ be a calculus. A proof for φ is a sequence of formulae $\varphi_0, \varphi_1, \ldots, \varphi_n$, $\varphi_i \in \text{WFF}_L$, $1 \leqslant i \leqslant n$, where $\varphi = \varphi_n$ and, for each φ_i, $\varphi_i \in \mathcal{A}$ or was obtained from $\{\varphi_0, \ldots, \varphi_{i-1}\}$ by an application of a rule in $\mathcal{R}$. If there is a proof for φ, then φ is a theorem.

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- Let $\varphi \in \text{WFF}_L$ and $\Gamma \subseteq \text{WFF}_L$. Let $\mathcal{C} = \langle \mathcal{A}, \mathcal{R} \rangle_L$ be a calculus. A proof for φ from Γ is a sequence of formulae $\varphi_0, \varphi_1, \ldots, \varphi_n$, $\varphi_i \in \text{WFF}_{\mathcal{L}}$, $1 \leqslant i \leqslant n$, where $\varphi = \varphi_n$ and, for each φ_i, $\varphi_i \in \mathcal{A} \cup \Gamma$ or was obtained from $\{\varphi_0, \ldots, \varphi_{i-1}\}$ by an application of a rule in $\mathcal{R}$. If there is a proof for φ from Γ, then φ is a demonstration.

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- Deduction theorem: let $\Gamma = \{\gamma_0, \ldots, \gamma_i\}$ for some $i \in \mathbb{N}$. If $\Gamma \vdash_{\mathcal{C}} \varphi$, then $\vdash_{\mathcal{C}} \gamma_0 \rightarrow (\ldots \rightarrow (\gamma_i \rightarrow \varphi))$.

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- Termination, convergence, etc.

Axiomatisation

Taut	enough propositional tautologies.
K	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$.

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SUB	Uniform substitution; and
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You can also add:

Dual	$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$

to the set of axioms, but it is not needed if you restrict the language to only one modal operator $\Box$ and take $\Diamond$ as an abbreviation.

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4. $\Box(p \wedge q) \to \Box p$ [MP,2,3]

Tableaux

- It is an analytic method: we “destroy” formulae when applying the inference rules. This means that termination (for some logics) is easy to achieve, as the obtained conclusions for each inference rule are “smaller” than its premisses.

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- We will consider that formulae are in Negation Normal Form: negation is applied only to propositional symbols, conjunctions and disjunctions are the only classical connectives allowed; for boxes and diamonds, we move negation inwards using the equivalences:

$$\neg\Box\varphi = \Diamond\neg\varphi \text{ and } \neg\Diamond\varphi = \Box\neg\varphi$$

Example - NNF

Suppose we want to prove:

$$\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$$

We start by negating it (because this is a method by contradiction):

$$\neg(\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q))$$

and proceed to transform it in its NNF:

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The Rules

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α	β	γ	δ
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This calculus is not confluent: you need to apply all the α and β rules before applying the δ rules. The γ rules should be applied last.

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Suppose we want to prove: $\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$:
We start by negating it: $\neg(\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q))$
and putting it into its NNF: $\Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q)$
Now we can start the tableaux construction:

(1) 1: $\Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q)$ [neg. assumption]

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X

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(6) 1.1: $\neg p$ $[\delta, 4]$ | (9) 1.2: $\neg q$ $[\delta, 5]$

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Example - II

Suppose we want to prove: $\Box(p \vee q) \to (\Box p \vee \Box q)$:
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and putting it into its NNF: $\Box(p \vee q) \wedge (\Diamond\neg p \wedge \Diamond\neg q)$
Now we can start the tableaux construction:

(1) 1: $\Box(p \vee q) \wedge (\Diamond\neg p \wedge \Diamond\neg q)$ [neg. assumption]

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Soundness

- We need to show that every step in the construction of the tableaux is satisfiability preserving.
- We have only four rules:

α	β	γ	δ
$\dfrac{\sigma : \varphi \wedge \psi}{\begin{array}{c}\sigma : \varphi \\ \sigma : \psi\end{array}}$	$\dfrac{\sigma : \varphi \vee \psi}{\sigma : \varphi \mid \sigma : \psi}$	$\dfrac{\sigma : \Box\varphi}{\sigma.i : \varphi}$ for all existing $\sigma.i$	$\dfrac{\sigma : \Diamond\varphi}{\sigma.i : \varphi}$ for a fresh $\sigma.i$

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- So, we need to prove that given a model, if it satisfies the premises, then it satisfies the conclusion.
- Let $\mathcal{M} = \langle \mathcal{W}, R, \pi \rangle$ be a model such that $\mathcal{M}$ satisfies $\sigma : \varphi \wedge \psi$.

Satisfaction of Prefixed Formulae

Definition 2.5.1 Let $\sigma : \varphi$ be a prefixed formula, where $\varphi \in$ WFF. Also, let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ be a Kripke structure. Finally, let Σ be a set of prefixes and $f : \Sigma \longrightarrow \mathcal{W}$ be a function that assigns to each prefix a world in $\mathcal{M}$ in such a way that:

- If σ and $\sigma.i$ are prefixes, then $f(\sigma)\mathcal{R}f(\sigma.i)$; and
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A tableau branch is satisfiable if its set of prefixed formulae is satisfiable. A tableau is satisfiable if it has a satisfiable branch.

Soundness again

Remember: if $\vdash_{\mathcal{C}} \varphi$, then $\mathcal{F} \models \varphi$
It is important to also remember that we are dealing with a refutational calculus. This means that if the formula we are dealing with is valid, then the tableau for its negation is closed. The first lemma says that there cannot be a model for a formula if its tableau is closed:

Proposition 2.5.2 A closed tableau is not satisfiable.

Proof (by contradiction). Let $\mathcal{T}$ be a closed tableau and assume it is satisfiable.

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The next lemma shows that any extension of a satisfiable tableau is satisfiable.

Proposition 2.5.3 Let $\mathcal{T}$ be a tableau and $\mathcal{T}'$ be the tableau obtained from $\mathcal{T}$ by an application of any of the inference rules. If $\mathcal{T}$ is satisfiable, then $\mathcal{T}'$ is also satisfiable.

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Soundness - Theorem

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Theorem 2.5.4 Let $\varphi \in$ WFF and $\mathcal{T}$ a closed tableau for φ. Then, φ is valid.

Just remember now that the closed tableau for φ starts with $1 : \neg\varphi$.

Proof: (By contradiction). Assume that φ is not valid. Then, by definition, there is a model $\mathcal{M}$ and a world w in $\mathcal{M}$ such that w does not satisfy φ. By the semantics of negation, $\mathcal{M}, w \models \neg\varphi$, for w in $\mathcal{M}$. This means that the set $\{1 : \neg\varphi\}$ is satisfiable. Take $\mathcal{M}$ as a model and let $f(1) = w$. By Proposition 2.5.3, all tableaux we might get for $\{1 : \neg\varphi\}$ are satisfiable. But, from Proposition 2.5.2, because $\mathcal{T}$ is closed, we know that this cannot happen. It follows that φ is valid.

Completeness

Definition 2.5.5 A tableau is saturated if no further rules can be applied.

Proposition (Page 61) All tableaux constructions are terminating.
Sketch: If the construction is *systematic*, this is easy to prove. We have

already defined a systematic construction: use α and β rules first; then apply δ; and finally apply γ. Note that all steps consist of adding subformulae to the tableau and the number of subformulae of a formula is finite.

Completeness - Continued

Theorem 2.5.7 Let $\varphi \in$ WFF. If φ is valid, then there is a closed tableau for φ.

Proof: We take the contrapositive: If φ has an open tableau, then φ is not valid.

Completeness - Continued

Theorem 2.5.7 Let $\varphi \in$ WFF. If φ is valid, then there is a closed tableau for φ.

Proof: We take the contrapositive: If φ has an open tableau, then φ is not valid. Assume that φ has an open (saturated) tableau $\mathcal{T}$. We show how to construct a model from this tableau. Take a branch $\mathcal{B}$ which is open in $\mathcal{T}$. Let $\mathcal{M}$ be as follows:

- $\mathcal{W} = \{\sigma \mid \sigma : \varphi \in \mathcal{B}\}$.
- if σ and $\sigma.i$ occur in $\mathcal{B}$, then set $\sigma \mathcal{R} \sigma.i$
- If $\sigma : p$, for $p \in \mathcal{P}$, occur in $\mathcal{B}$, then set $\pi(\sigma, p) = \text{true}$; otherwise $\pi(\sigma, p) = \text{false}$.

This construction is correct, that is, the built model is indeed a model for $\mathcal{B}$.

Example: $\Box(p \lor q) \to (\Box p \lor \Box q)$

Negate: $\neg(\Box(p \lor q) \to (\Box p \lor \Box q))$. NNF: $\Box(p \lor q) \land (\Diamond\neg p \land \Diamond\neg q)$

(1) 1: $\Box(p \lor q) \land (\Diamond\neg p \land \Diamond\neg q)$ [neg. assumption]

(2) 1: $\Box(p \lor q)$ $[\alpha, 1]$

(3) 1: $\Diamond\neg p \land \Diamond\neg q$ $[\alpha, 1]$

(4) 1: $\Diamond\neg p$ $[\alpha, 3]$

(5) 1: $\Diamond\neg q$ $[\alpha, 3]$

(6) 1.1: $\neg p$ $[\delta, 4]$

(7) 1.1: $p \lor q$ $[\gamma, 2]$

(8) 1.1: p $[\beta, 7]$ (9) 1.1: q $[\beta, 7]$

X

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X

(10) 1.2: $\neg q$ $[\delta, 5]$

(11) 1.2: $p \lor q$ $[\gamma, 2]$

(12) 1.2: p $[\beta, 11]$ (13) 1.2: q $[\beta, 11]$

X

Example: $\Box(p \vee q) \to (\Box p \vee \Box q)$

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$\mathcal{W} = \{1, 1.1, 1.2\}$

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$$\pi(1, p) = \pi(1, q) = \text{false}$$

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$$\mathcal{M}, 1 \models \Box(p \vee q), \mathcal{M}, 1 \not\models \Box p \vee \Box q$$

This is just one part of the story...

... better: this is just part one of the story...

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What do you need to extend the calculi for multimodal logics?

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We start with the easy part: the axiomatic system.

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and

SUB Uniform substitution; and

MP If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$.

Nec If $\vdash \varphi$, then $\vdash \boxed{a}\varphi$

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Tableaux for multimodal logics

α	β	γ	δ
$\dfrac{\sigma : \varphi \wedge \psi}{\begin{array}{c}\sigma : \varphi \\ \sigma : \psi\end{array}}$	$\dfrac{\sigma : \varphi \vee \psi}{\sigma : \varphi \mid \sigma : \psi}$	$\dfrac{\sigma : \Box \varphi}{\sigma.i : \varphi}$ for all existing $\sigma.i$	$\dfrac{\sigma : \Diamond \varphi}{\sigma.i : \varphi}$ for a fresh $\sigma.i$

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Example

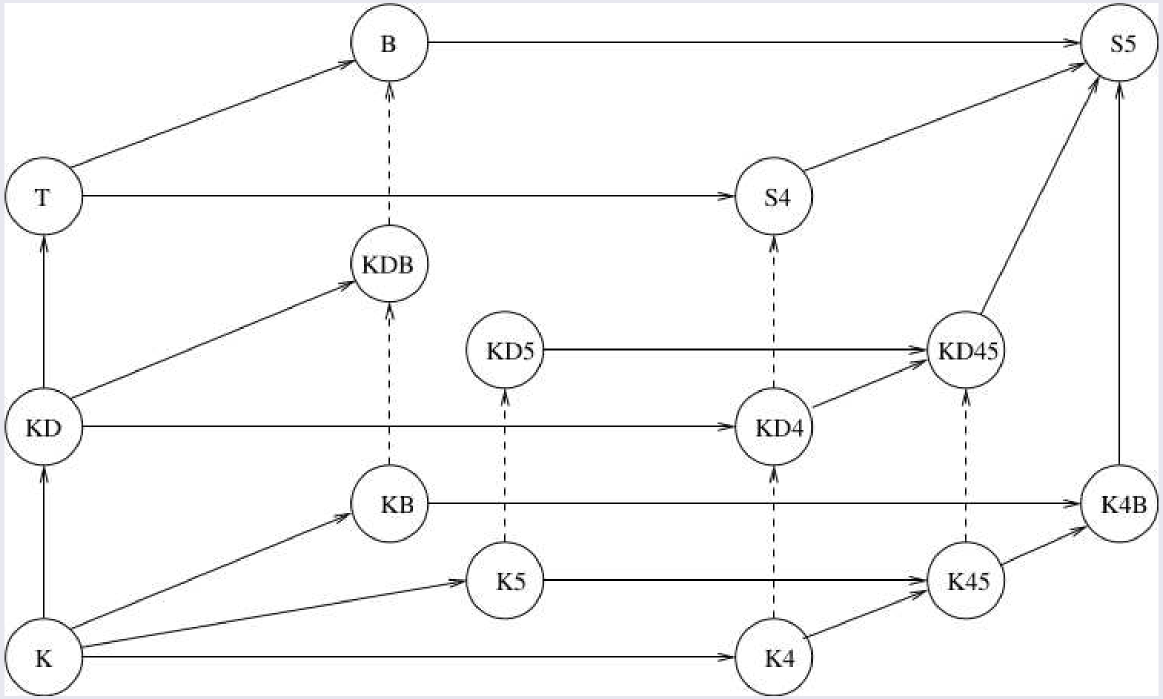
$$\Diamond_1(\Diamond_2 p \vee \Diamond_2 q) \wedge (\Box_1\Box_2 p \vee \Box_1\Box_2 q)$$

To be continued.

Some Other Usual Modal Logics

Different restrictions on the accessibility relations $\mathcal{R}_a$ define different modal logics:

- No restrictions: K_n;
- Reflexive: KT_n;
- Transitive: $\mathsf{K4}_n$;
- Euclidean: $\mathsf{K5}_n$;
- Serial: KD_n;
- Symmetric: KB_n;
- Reflexive and Transitive: $\mathsf{S4}_n$;
- Reflexive and Euclidean: $\mathsf{S5}_n$;
-



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