Modal Logic: Overview

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LMU, The Modal Logic Sessions

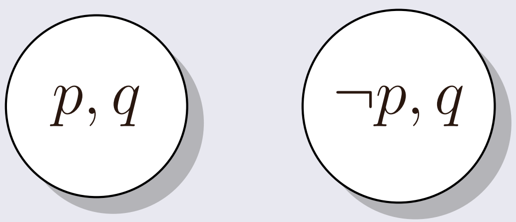
The Basics

Motivation

- Modal logics have been used in Computer Science to represent properties of complex systems: temporal, epistemic, obligations, choice, actions, and so on.
- Applications include, but are not restricted to: programming languages, knowledge representation and reasoning, verification of distributed systems and terminological reasoning.

Syntax and Semantics

- Modal logics are extensions of propositional logic with operators '$\Box$' and '$\Diamond$'.
- Evaluation of a formula depends on a set of worlds and on the accessibility relations on this set.
- Different restrictions on the accessibility relations give rise to different modal logics.



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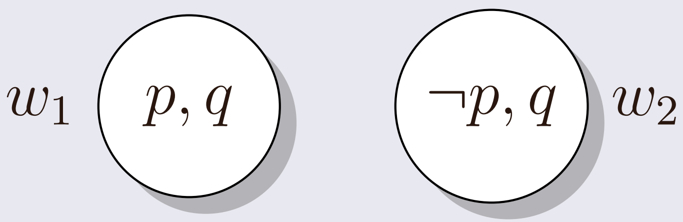

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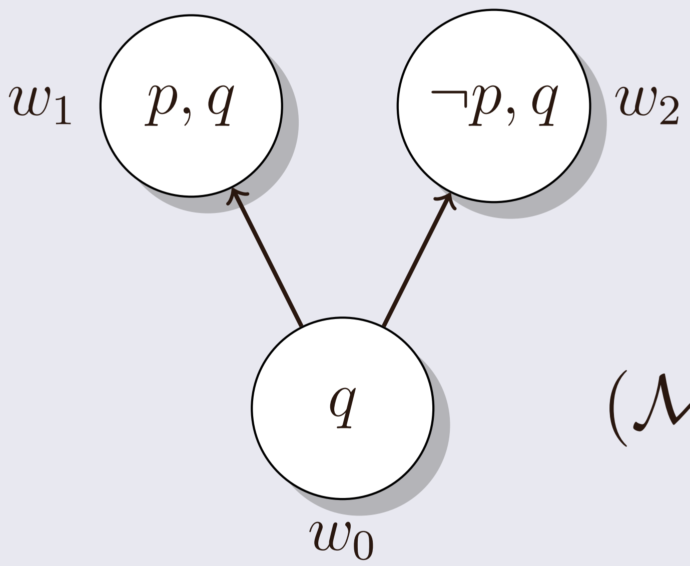

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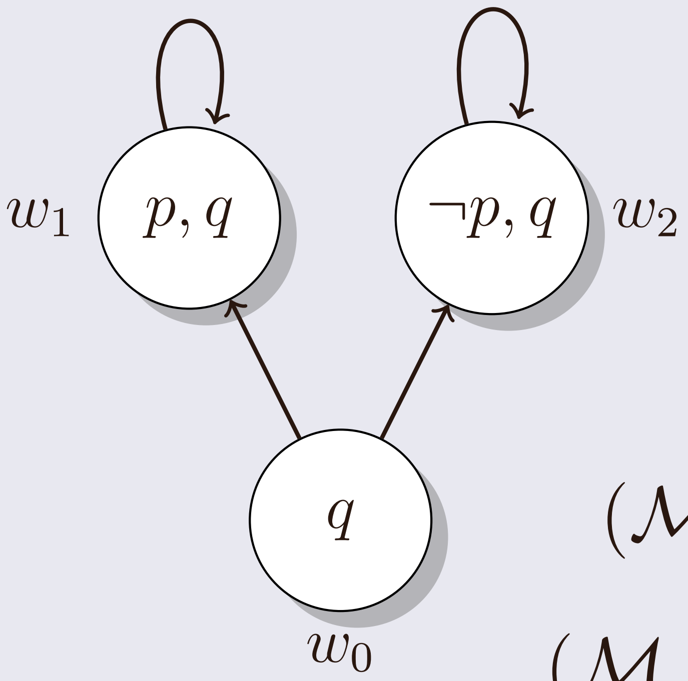

$$(\mathcal{M}, w_0) \models \Diamond p \land \Diamond \neg p$$

$$(\mathcal{M}, w_0) \models \Box(\Box p \lor \Box \neg p)$$

Syntax and Semantics

- Modal logics are extensions of propositional logic with operators '$\boxed{a}$' and '$\langle a \rangle$', where $a \in \mathcal{A} = \{1, \ldots, n\}$, $n \in \mathbb{N}$.
- Evaluation of a formula depends on a set of worlds and on the accessibility relations on this set.
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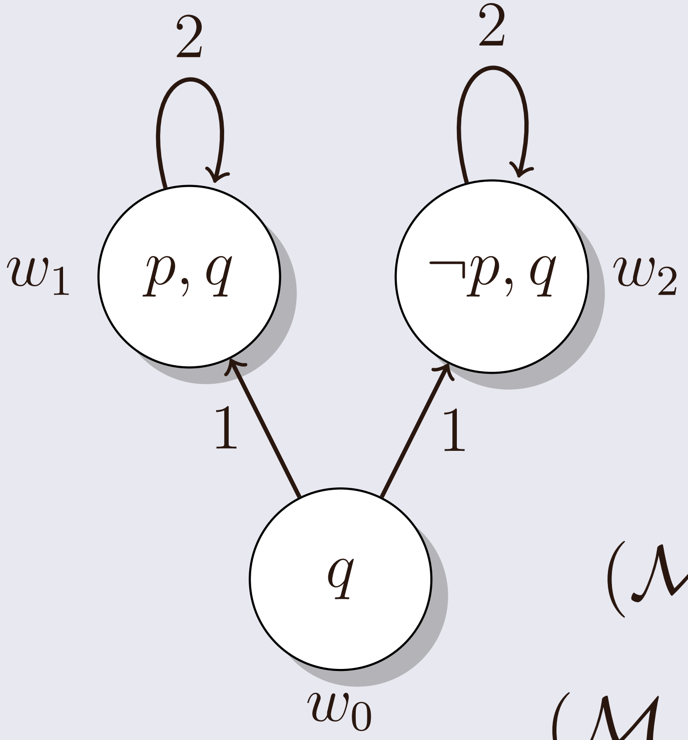

$$(\mathcal{M}, w_0) \models \langle 1 \rangle p \wedge \langle 1 \rangle \neg p$$

$$(\mathcal{M}, w_0) \models \boxed{1}(\boxed{2} p \vee \boxed{2} \neg p)$$

Relations

- If p is an author, then p wrote a paper:

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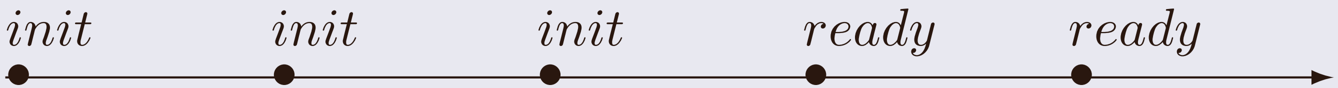
- Having an ancestor is a transitive relation:

$$< has\_ancestor > \top \rightarrow < has\_ancestor >< has\_ancestor > \top$$


sonnig
neblig
regnerisch
regnerisch
sonnig


$init$
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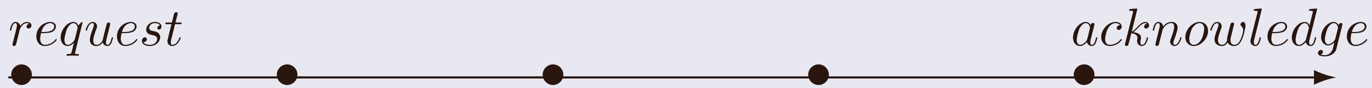
More on Computational Systems



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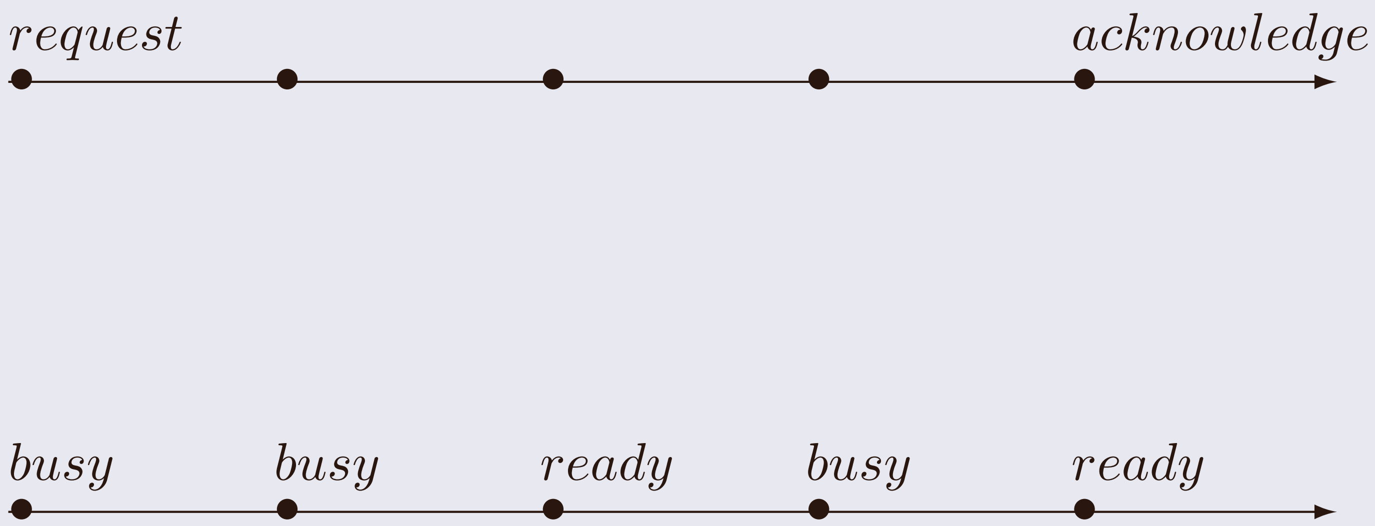

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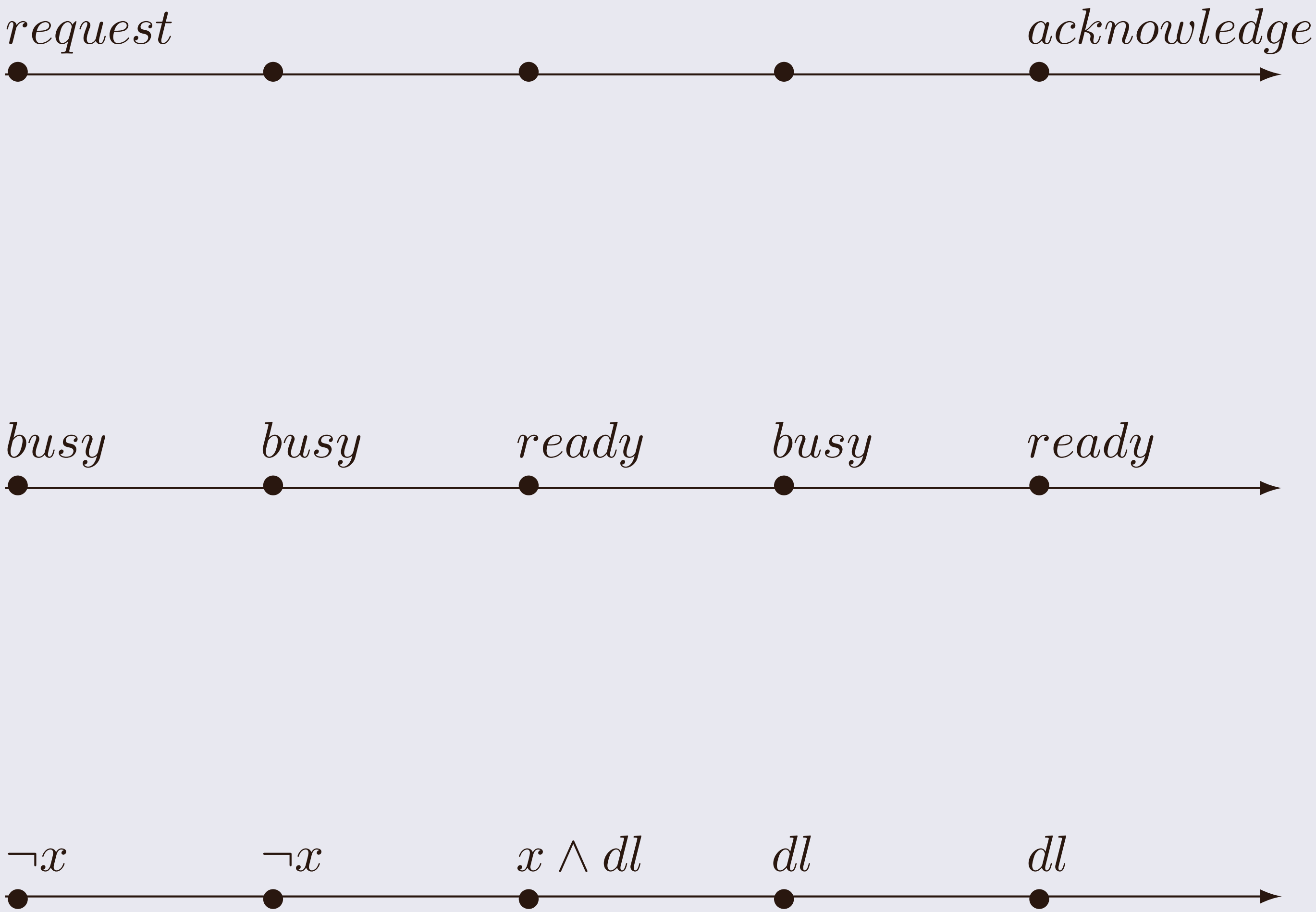
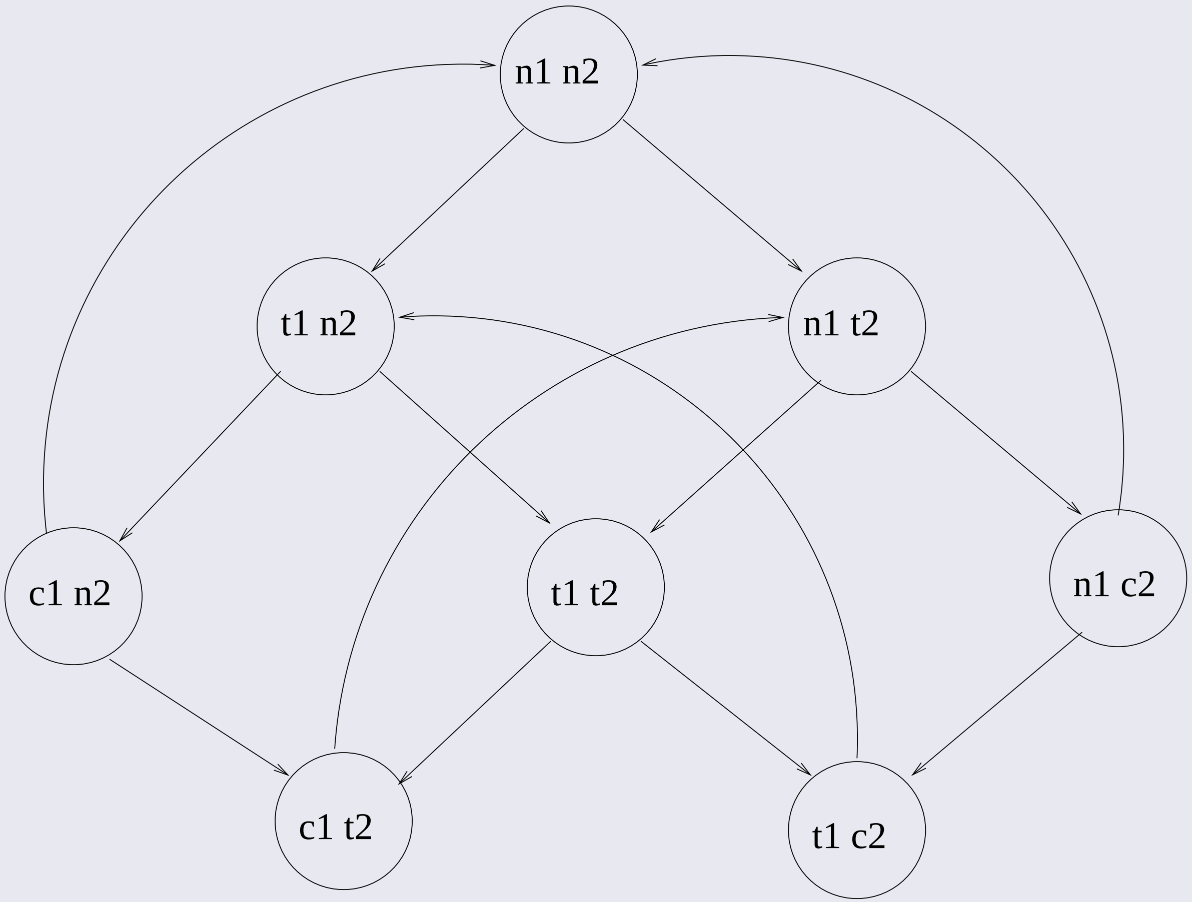


Mutual Exclusion



Syntax

- The set of well-formed formulae, WFF:
  - $p \in \mathcal{P}$;
  - if $\varphi \in$ WFF, then so are $\neg\varphi$ and $\boxed{a}\varphi$, $a \in \mathcal{A} = \{1, \ldots, n\}$;
  - if φ and $\psi \in$ WFF, then $(\varphi \wedge \psi) \in$ WFF.
- Abbreviations:
  - $\text{false} \equiv p \wedge \neg p$ (for $p \in \mathcal{P}$)
  - $\text{true} \equiv \neg\text{false}$
  - $\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$
  - $\varphi \to \psi \equiv \neg\varphi \vee \psi$
  - $\varphi \leftrightarrow \psi \equiv (\varphi \to \psi) \wedge (\psi \to \varphi)$
  - $\Diamond_a\varphi \equiv \neg\boxed{a}\neg\varphi$.

Semantics

- A Kripke Structure $\mathcal{M}$ for $\mathcal{P}$ and $\mathcal{A} = \{1, \ldots, n\}$ is a tuple

$$\mathcal{M} = \langle \mathcal{W}, \mathcal{R}_1, \ldots, \mathcal{R}_n, \pi \rangle,$$

  where:

  - $\mathcal{W}$ is a non-empty set;
  - For each $a \in \mathcal{A}$, $\mathcal{R}_a \subseteq \mathcal{W} \times \mathcal{W}$;
  - $\pi : \mathcal{W} \times \mathcal{P} \longrightarrow \{T, F\}$.

- The satisfiability relation $\models$ between a world $w \in \mathcal{W}$ in a Kripke structure $\mathcal{M}$ and a formula is inductively defined by:

  - $(\mathcal{M}, w) \models p$, $p \in \mathcal{P}$, iff $\pi(w, p) = T$;
  - $(\mathcal{M}, w) \models \neg\varphi$ iff $(\mathcal{M}, w) \not\models \varphi$;
  - $(\mathcal{M}, w) \models \varphi \wedge \psi$ iff $(\mathcal{M}, w) \models \varphi$ and $(\mathcal{M}, w) \models \psi$;
  - $(\mathcal{M}, w) \models \boxed{a}\varphi$ iff for all w', $w\mathcal{R}_a w'$ implies $(\mathcal{M}, w') \models \varphi$;
  - $(\mathcal{M}, w) \models \Diamond_a\varphi$ iff exists w', $w\mathcal{R}_a w'$ and $(\mathcal{M}, w') \models \varphi$.

Reasoning Tasks

$$\mathcal{M} = \langle \mathcal{W}, \mathcal{R}_1, \ldots, \mathcal{R}_n, \pi \rangle$$

- A formula φ is locally satisfiable iff there is a model $\mathcal{M}$ and $w \in \mathcal{W}$ such that $\langle \mathcal{M}, w \rangle \models \varphi$. In this case, we say that $\mathcal{M}$ satisfies φ, denoted by $\mathcal{M} \models_L \varphi$.

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- A formula φ is satisfiable under the global constraints $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ iff there is a model $\mathcal{M}$ such that $\mathcal{M} \models_G \Gamma$ and there is $w \in \mathcal{W}$ such that $\langle \mathcal{M}, w \rangle \models_L \varphi$.

Reasoning Tasks

$$\mathcal{M} = \langle \mathcal{W}, \mathcal{R}_1, \ldots, \mathcal{R}_n, \pi \rangle$$

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  PSPACE-complete [Ladner, 1977, Halpern and Moses, 1992]

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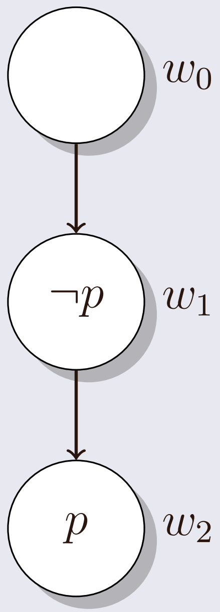
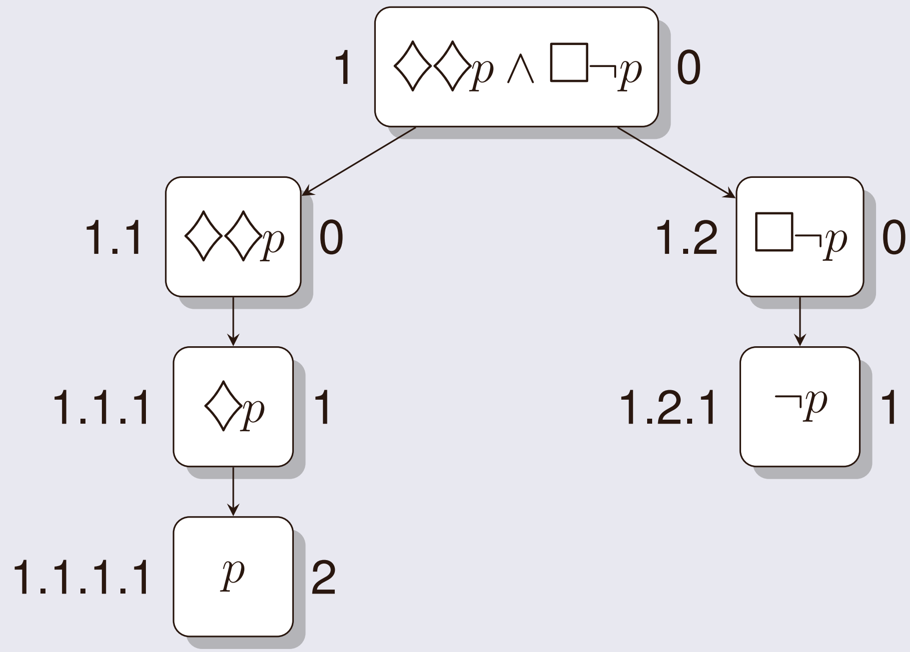
Local Reasoning

- Nice properties: finite, tree-like models with height bounded by the modal depth/modal level of the formula.

$$\Diamond\Diamond p \wedge \Box\neg p$$





Invariance Results

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What properties are preserved by relations and operations. Two models are modally equivalent if they have the same theories, i.e. for $\mathcal{M}_1$ and $\mathcal{M}_2$, and all formulae φ, we have that $\mathcal{M}_1 \equiv \mathcal{M}_2$ if, and only if,

$$\mathcal{M}_1 \models \varphi \text{ if, and only if, } \mathcal{M}_2 \models \varphi.$$

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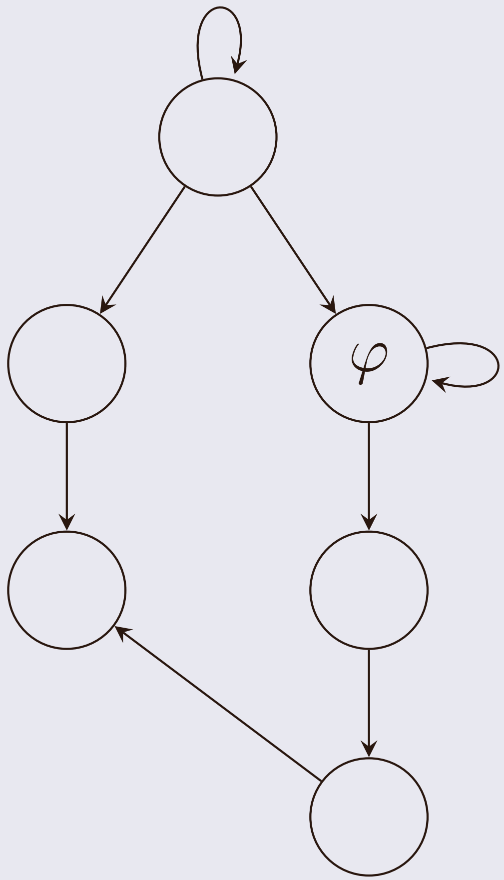
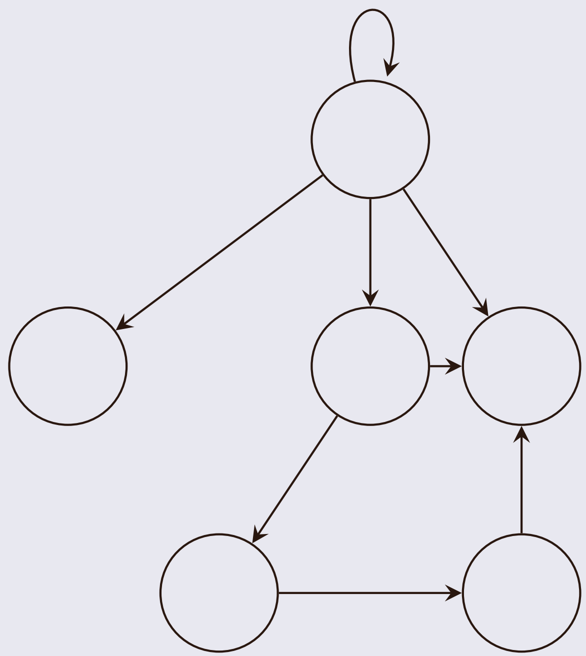
- disjoint unions
- generated submodels
- bounded morphisms
- n-bissimilarity

These results imply that if a formula is satisfiable, then it is satisfiable in a tree-like finite model.

Proofs in Pictures





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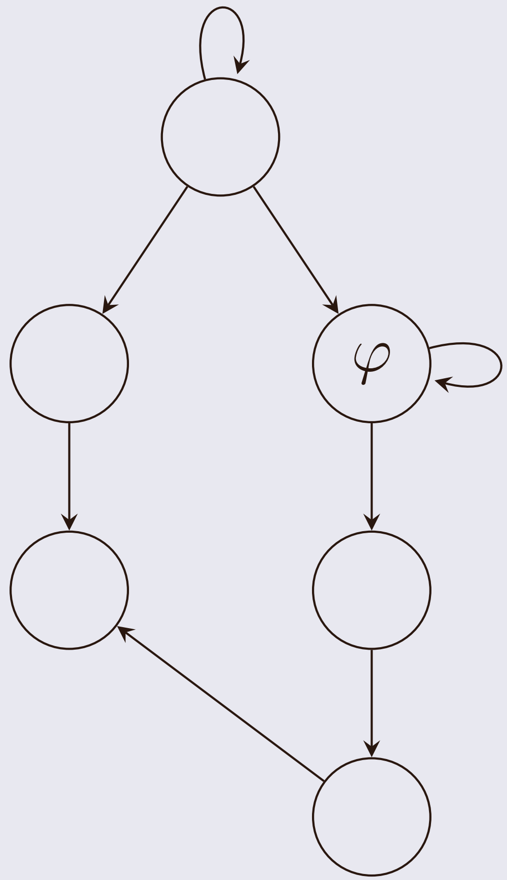

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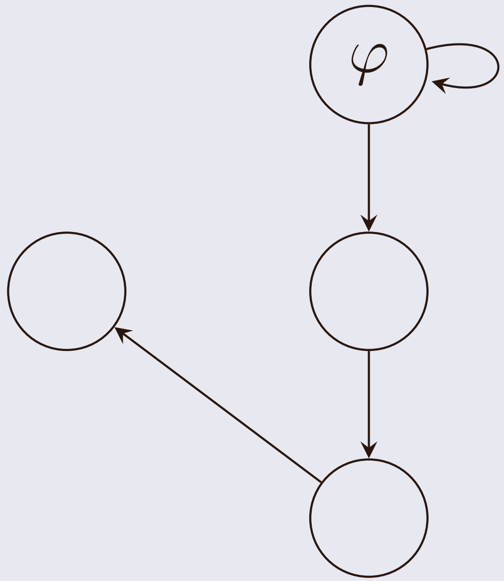

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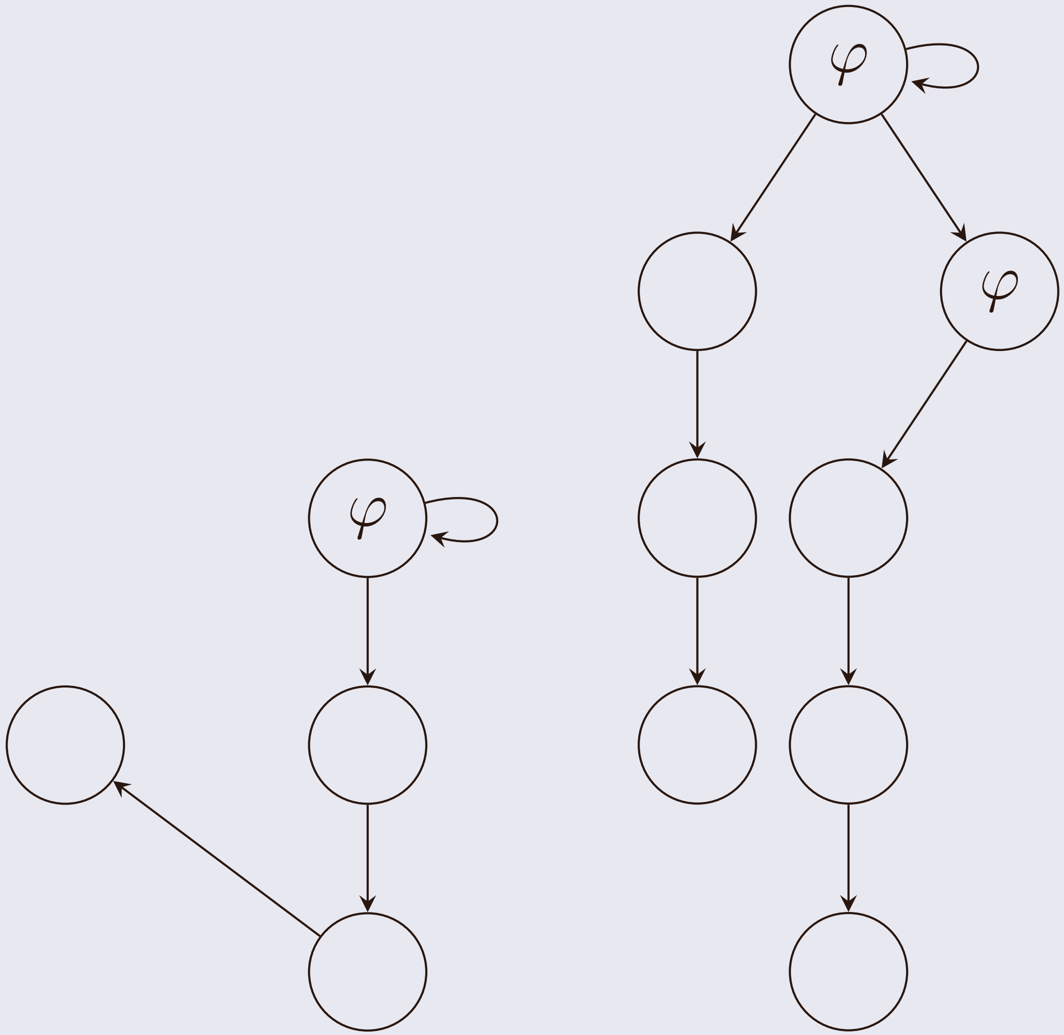

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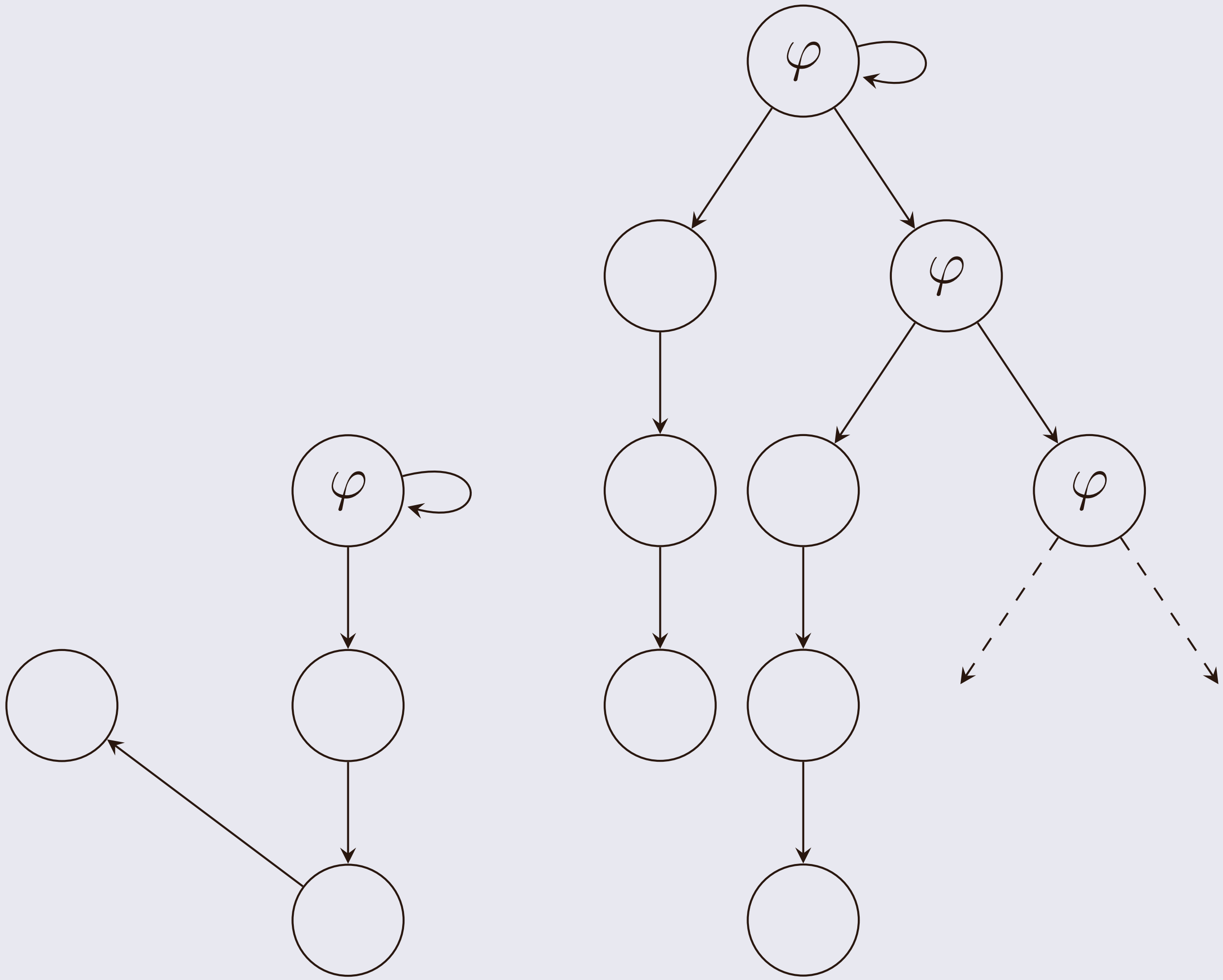


Modal Depth

Definition 2.28 Let $\varphi, \psi, \chi \in$ WFF be formulae. The modal depth (or degree) of φ is the maximum nesting of modal operator occurring in φ. Let deg : WFF $\longrightarrow \mathbb{N}$ be a function defined as follows:

- $deg(\varphi) = 0$, if $\varphi \in \mathcal{P}$
- $deg(\bot) = 0$
- $deg(\neg\varphi) = deg(\varphi)$
- $deg(\varphi \wedge \psi) = \max\{deg(\varphi), deg(\psi)\}$
- $deg(\Box\varphi) = 1 + deg(\varphi)$

Example, with boxes

$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$$

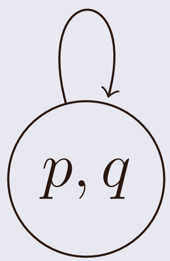
Example, with boxes

$$\Box(p \to q) \to (\Box p \to \Box q)$$

The modal degree of this formula is one; thus, we only need to look at trees of height 1 (note, heights start at 0).

$$\Diamond(p \wedge \neg q) \vee \Diamond\neg p \vee \Box q$$

is satisfied at $\mathcal{M}$:



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$$\Diamond(p \land \neg q) \lor \Diamond\neg p \lor \Box q$$

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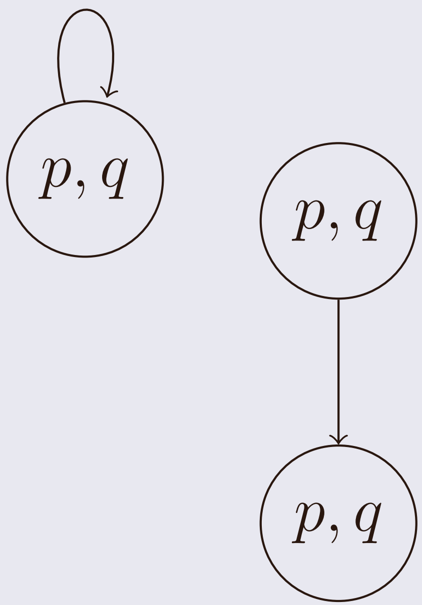

The next sessions

- Axiomatic systems
- Tableaux

Some Other Usual Modal Logics

Different restrictions on the accessibility relations $\mathcal{R}_a$ define different modal logics:

- No restrictions: K_n;
- Reflexive: KT_n;
- Transitive: $\mathsf{K4}_n$;
- Euclidean: $\mathsf{K5}_n$;
- Serial: KD_n;
- Symmetric: KB_n;
- Reflexive and Transitive: $\mathsf{S4}_n$;
- Reflexive and Euclidean: $\mathsf{S5}_n$;
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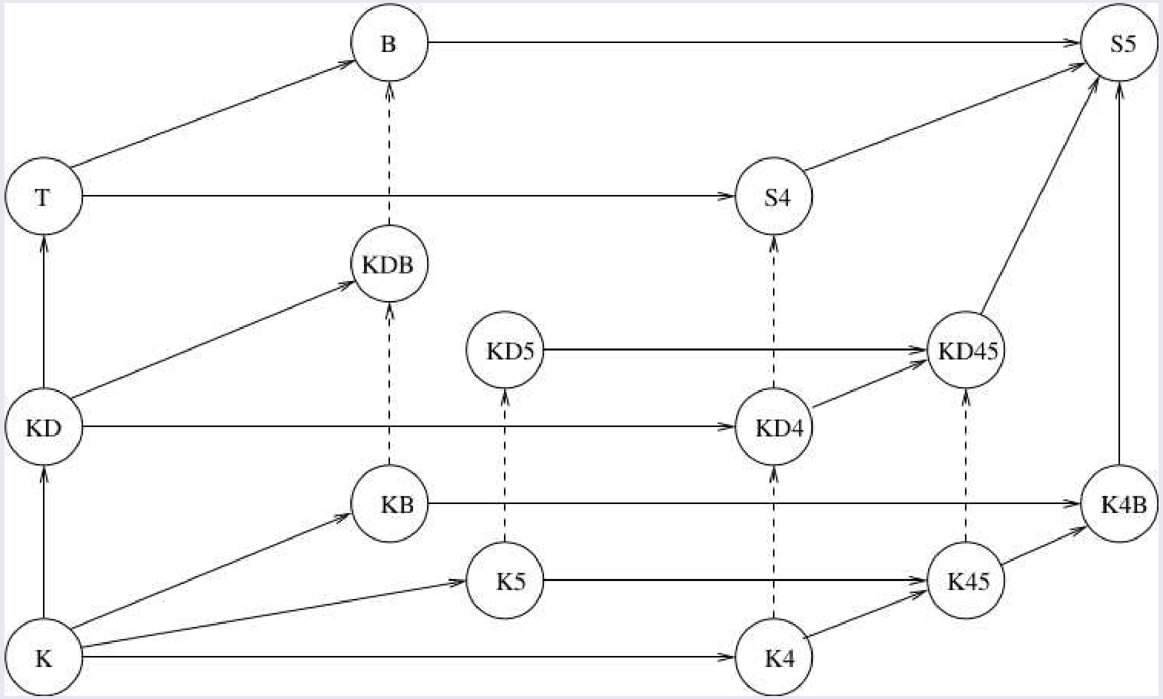

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Invariance in Detail

Disjoint Unions

Let $\mathcal{M}_i = \langle \mathcal{W}_i, \mathcal{R}_i, \pi_i \rangle$, $i \in \mathbb{N}$, be Kripke structures. The disjoint union of $\mathcal{M}_i$ is given by:

$$\biguplus_i \mathcal{M}_i = \langle \bigcup_i \mathcal{W}_i, \bigcup_i \mathcal{R}_i, \bigcup_i \pi_i \rangle$$

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$$\biguplus_i \mathcal{M}_i = \langle \bigcup_i \mathcal{W}_i, \bigcup_i \mathcal{R}_i, \bigcup_i \pi_i \rangle$$

Proposition 2.3 Modal satisfaction is invariant under disjoint unions:

$$\mathcal{M}_i \models \varphi \text{ if, and only if, } \biguplus_i \mathcal{M}_i \models \varphi.$$

Generated Submodels

Definition 2.5, submodel: Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ be Kripke structures. If

- $\mathcal{W}' \subseteq \mathcal{W}$,
- $\mathcal{R}' = \mathcal{R} \cap (\mathcal{W}' \times \mathcal{W}')$, and
- $\pi'(w, p) = \pi(w, p)$,

for all $w \in \mathcal{W}'$ and $p \in \mathcal{P}$, then $\mathcal{M}'$ is a submodel of $\mathcal{M}$.

Definition 2.5, generated submodel: Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ be a Kripke structure and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ a submodel of $\mathcal{M}$. We say that $\mathcal{M}'$ is a generated submodel of $\mathcal{M}$, if: $w \in \mathcal{W}'$ and $w\mathcal{R}v,$ then $v \in \mathcal{W}'$.

Definition 2.5, generated submodel by a set: Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ be a Kripke structure. A submodel generated by $\mathcal{W}''$ is the smallest generated submodel $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ such that $\mathcal{W}'' \subseteq \mathcal{W}'$.

Definition 2.5, rooted (or pointed) generated model: Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ be a Kripke structure. A rooted generated model, with root $w \in \mathcal{W}$, is a submodel generated by $\{w\}$.

Proposition 2.6 Modal satisfaction is invariant under generated submodels, that is, if $\mathcal{M}'$ is a generated submodel of $\mathcal{M}$, then:

$$\mathcal{M} \models \varphi \text{ if, and only if, } \mathcal{M}' \models \varphi$$

for all $\varphi \in$ WFF.

Homomorphism

Definition 2.7, homomorphism Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ be Kripke structures, and $f : \mathcal{M} \longrightarrow \mathcal{M}'$ be a function. If:

- for all $p \in \mathcal{P}$ and $w \in \mathcal{W}$, if $\pi(w)(p) =$ *true*, then $\pi'(f(w), p) =$ *true*; and
- if $w\mathcal{R}w'$, then $f(w)\mathcal{R}'f(w')$;

then, f is homomorphism from $\mathcal{M}$ to $\mathcal{M}'$.
Note that this is not enough show invariance.

Homomorphism

Definition 2.8, strong homomorphism: Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ be Kripke structures, and $f : \mathcal{M} \longrightarrow \mathcal{M}'$ be a function. If:

- for all $p \in \mathcal{P}$ and $w \in \mathcal{W}$, $\pi(w)(p) =$ *true* if, and only if, $\pi'(f(w), p) =$ *true*; and
- $w\mathcal{R}w'$ if, and only if, $f(w)\mathcal{R}'f(w')$;

then, f is strong homomorphism from $\mathcal{M}$ to $\mathcal{M}'$.

Definition 2.8, embedding: Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ be Kripke structures, and $f : \mathcal{M} \longrightarrow \mathcal{M}'$ be a strong homomorphism. If f is injective, then f is an embedding.

Definition 2.8, isomorphism: Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ be Kripke structures, and $f : \mathcal{M} \longrightarrow \mathcal{M}'$ be a strong homomorphism. If f is bijective, then f is an isomorphism.

Invariance Results

Proposition 2.9 Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ be Kripke structures, and $f : \mathcal{M} \longrightarrow \mathcal{M}'$ be a function. The following holds:

1. if f is a surjective strong homomorphism with $f(w) = w'$, then w and w' are modally equivalent.
2. if $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic, then they are modally equivalent.

Bounded Morphism

Definition 2.10, bounded morphism Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ be Kripke structures, and $f : \mathcal{M} \longrightarrow \mathcal{M}'$ a function. If:

1. w and $f(w)$ satisfy the same propositional symbols; that is, for all $w \in \mathcal{W}$ and $p \in \mathcal{P}$:

$$\pi(w, p) = \pi'(f(w), p)$$

2. for all $w, w' \in \mathcal{W}$, if $w\mathcal{R}w'$, then $f(w)\mathcal{R}'f(w')$;
3. if $f(w)\mathcal{R}'w'$, then there is $w'' \in \mathcal{W}$ such that $w\mathcal{R}w''$ and $f(w'') = w'$ (back condition);

then f is a bounded morphism from $\mathcal{M}$ to $\mathcal{M}'$.

Definition 2.10, bounded morphic image Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ be Kripke structures, and $f : \mathcal{M} \longrightarrow \mathcal{M}'$ a bounded morphism from $\mathcal{M}$ to $\mathcal{M}'$. If f is surjective, then $\mathcal{M}'$ is a bounded morphic image of $\mathcal{M}$.

Proposition 2.14 Modal satisfaction is invariant under bounded morphisms.
Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ be Kripke structures, and $f : \mathcal{M} \longrightarrow \mathcal{M}'$ a bounded morphism from $\mathcal{M}$ to $\mathcal{M}'$. Then,

$$\mathcal{M} \models \varphi \text{ if, and only if, } \mathcal{M}' \models \varphi$$

for all $\varphi \in$ WFF.

Tree Model Property

Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ be a Kripke structure. $\mathcal{M}$ is tree-like if the graph $\langle \mathcal{W}, \mathcal{R} \rangle$ is a tree (a directed acyclic graph).

Proposition 2.15 Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ be a Kripke structure. Then, there is $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ that is tree-like and a bounded morphic image of $\mathcal{M}'$.

From the previous results, we have that all modal formulae that are satisfiable are satisfiable in a tree-like model.

Definition 2.16, bissimulations Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ be Kripke structures, and $\mathcal{Z} \subseteq \mathcal{M} \times \mathcal{M}'$ a relation. If:

1. if $w\mathcal{Z}w'$, then for all $w \in \mathcal{W}$ and $p \in \mathcal{P}$:

$$\pi(w, p) = \pi'(w', p)$$

2. if $w\mathcal{Z}w'$ and $w\mathcal{R}w''$, then there is $w''' \in \mathcal{W}'$ such that $w''\mathcal{Z}w'''$ and $w'\mathcal{R}'w'''$ (forth condition);
3. if $w\mathcal{Z}w'$ and $w'\mathcal{R}'w'''$, then there is $w'' \in \mathcal{W}$ such that $w''\mathcal{Z}w'''$ and $w\mathcal{R}w''$ (back condition);

then $\mathcal{Z}$ is a bissimulation between $\mathcal{M}, w$ and $\mathcal{M}', w'$.

Invariance Results

Proposition 2.19 Let $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{M}_i$ $(i \in \mathbb{N})$ be Kripke structures. The following holds:

1. if $\mathcal{M}$ and $\mathcal{M}'$ are modally equivalent, then they are bissimilar.
2. for all i, $w \in \mathcal{M}_i$, $\mathcal{M}_i, w$ is bissimilar to $\biguplus_i \mathcal{M}_i, w$.
3. if $\mathcal{M}'$ is a generated submodel of $\mathcal{M}$, then $\mathcal{M}', w$ is bissimilar to $\mathcal{M}, w$, for all $w \in \mathcal{M}'$.
4. if $\mathcal{M}'$ is a bounded morphic image of $\mathcal{M}$, then $\mathcal{M}', w$ is bissimilar to $\mathcal{M}, w$, for all $w \in \mathcal{M}'$.

Theorem 2.20 Let $\mathcal{M}$ and $\mathcal{M}'$ be Kripke structures. Then, for all $w \in \mathcal{W}$ and $w' \in \mathcal{W}'$,

if w and w' are bissimilar, then they are modally equivalent.

Note: the converse is not true in general; however, if the relations are finite, then the bissimilarity and modal equivalence coincide (Theorem 2.24, Hennessy-Milner Theorem).

Definition 2.7 Let $\mathfrak{M}$ be a class of Kripke structures and $\mathcal{L}$ a logical language. If, for all formulae $\varphi \in \text{WFF}_{\mathcal{L}}$:

if φ is satisfiable, then there is $\mathcal{M} \in \mathfrak{M}$, $\mathcal{M}$ finite, such that $\mathcal{M} \models \varphi$,

then $\mathcal{L}$ has the finite model property with respect to $\mathfrak{M}$.

Definition 2.28 Let $\varphi, \psi, \chi \in$ WFF be formulae. The modal depth (or degree) of φ is the maximum nesting of modal operator occurring in φ. Let deg : WFF $\longrightarrow \mathbb{N}$ be a function defined as follows:

- $deg(\varphi) = 0$, if $\varphi \in \mathcal{P}$
- $deg(\bot) = 0$
- $deg(\neg\varphi) = deg(\varphi)$
- $deg(\varphi \wedge \psi) = \max\{deg(\varphi), deg(\psi)\}$
- $deg(\Box\varphi) = 1 + deg(\varphi)$

Proposition 2.29 Assume $\mathcal{P}$ is finite.

1. For all $n \in \mathbb{N}$, there are only finitely many formulae of degree at most n (up to logical equivalence);
2. For all $n \in \mathbb{N}$, Kripke structures $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$, and $w \in \mathcal{W}$, $\{\varphi \mid \mathcal{M}, w \models \varphi, deg(\varphi) \leq n\}$ is finite (up to logical equivalence).

Definition 2.30 Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \pi \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \pi' \rangle$ be Kripke structures with $w \in \mathcal{W}$ and $w' \in \mathcal{W}'$. Let $\mathcal{Z}_i \subseteq \mathcal{W} \times \mathcal{W}'$, $i \leq n$, $n \in \mathbb{N}$, be relations such that $\mathcal{Z}_i \subseteq \mathcal{Z}_{i-1}$. If:

1. $w \mathcal{Z}_n w'$
2. for all $w \in \mathcal{W}$, if $w \mathcal{Z}_0 w'$, then for all $p \in \mathcal{P}$:

$$\pi(w, p) = \pi'(w', p)$$

3. for all $w, w' \in \mathcal{W}$, if $w \mathcal{Z}_{i+1} w'$ and $w \mathcal{R} w''$, then there exists w''' with $w' \mathcal{R}' w'''$ and $w'' \mathcal{Z}_i w'''$;
4. for all $w, w' \in \mathcal{W}$, if $w \mathcal{Z}_{i+1} w'$ and $w' \mathcal{R}' w'''$, then there exists w'' with $w \mathcal{R} w''$ and $w'' \mathcal{Z}_i w'''$;

then $\mathcal{M}, w$ and $\mathcal{M}', w'$ are n-bissimilar.

Proposition 2.31 n-bissimilarity for all n and modal equivalence coincide.

$\mathcal{M}, w$ and $\mathcal{M}', w'$ are n-bissimilar

if, and only if,

for all φ, $deg(\varphi) \leq n$ and $\mathcal{M}, w \models \varphi$ if, and only if, $\mathcal{M}', w' \models \varphi$.

Finite Model Property via Selection

Definition 2.32,height of a tree-like model It is defined as the height of trees.

Definition 2.32, restriction to a particular height k: Just take $\mathcal{W}'$ to be the set of worlds that occurr up to the height k.

Lemma 2.33 Worlds w in the model whose height is restricted by k are l-bissimilar to those in the original model, where $l = k - height(w)$.

Note: l-bissimilarity says that we are considering the formulae with degree at most l, that is, at the height $k - height(w)$ we are considering modal formulae with degree at most $k - height(w)$. Taking k to be the degree of a formula φ, this says that the subformulae of φ are satisfied at the height they occur in the tree.

Continued

We take a formula that is satisfiable and show that it is satisfiable in a model that is restricted by some height k.

1. Take the tree-like model with root w that satisfies the formula (Proposition 2.6, the unravelling construction, bounded morphism).
2. Take the model restricted by k, the degree of the formula (Lemma 2.33).
3. The construction basically divides the tree in layers (corresponding to the sets of worlds that are at some height)
   (a) $S_0 = \{w\}$
   (b) $S_{i+1} = \{w' \mid w \in S_i, w \models \Diamond\psi, deg(\Diamond\psi) = i, w' \models \psi, \}$

By Propositon 2.29, there are only finitely many formulae of the form $\Diamond\psi$ (up to equivalence) whose degree is i.

Filtrations

1. It works on the closed set of subformulae Σ.
2. Defines equivalence classes for worlds based on the formulae in Σ.
3. Construct the model using those equivalence classes and making sure that diamonds are satisfied.

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3. Construct the model using those equivalence classes and making sure that diamonds are satisfied.

Example:

$$\mathcal{M} = \langle \mathbb{N}, \{(0,1),(0,2),(1,3)\} \cup \{(n,n+1) \mid n \geq 2\}, \pi \rangle$$

where $\pi(w,p) =$ *true* iff $w \neq 0$ and $\pi(w,q) =$ *true* iff $w = 2$.
Take $\Sigma = \{p, \Diamond p\}$. There are only two equivalence classes based on Σ for this particular model: those that satisfy $\{p\}$ and those that don't.

$$\mathcal{M}' = \langle \{|0|,|1|\}, \{(|0|,|1|),(|1|,|1|)\}, \pi' \rangle$$

where $\pi(w,p) =$ *true* iff $w = |1|$.

Proposition 2.38 The construction of the filtration is finite: it has the size of the powerset of Σ.

Theorem 2.39 Satisfiability of modal formula is preserved under filtration.

Smallest and largest filtrations can be obtained by restricting the construction of the relation in the filtered model (Lemma 2.40):

1. $|w|\mathcal{R}^s|w'|$ iff exists $w'' \in |w|$, $w''' \in |w'|$ and $w''\mathcal{R}w'''$.
2. $|w|\mathcal{R}^l|w'|$ iff for all $\Diamond\varphi$ in Σ, if $\mathcal{M}, w' \models \psi$, then $\mathcal{M}, w \models \Diamond\varphi$.

Theorem 2.41 If a formula is satisfiable, it is satisfiable in a finite model.

Proof: using a filtration, the size of the model is at most exponential in the number of subformulae.